

Acutely Triangulated, Stacked, and Very Ununfoldable Polyhedra

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Abstract

We present new examples of topologically convex edge-ununfoldable polyhedra, i.e., polyhedra that are combinatorially equivalent to convex polyhedra, yet cannot be cut along their edges and unfolded into one planar piece without overlap. One family of examples is *acutely triangulated*, i.e., every face is an acute triangle. Another family of examples is *stacked*, i.e., the result of face-to-face gluings of tetrahedra. Both families achieve another natural property, which we call *very unfoldable*: for every k , there is an example such that every nonoverlapping multipiece edge unfolding has at least k pieces.

1 Introduction

Can every convex polyhedron be cut along its edges and unfolded into a single planar piece without overlap? Such *edge unfoldings* or *nets* are useful for constructing 3D models of a polyhedron (from paper or other material such as sheet metal): cut out the net, fold along the polyhedron’s uncut edges, and re-attach the polyhedron’s cut edges [25]. Unfoldings have also proved useful in computational geometry algorithms for finding shortest paths on the surface of polyhedra [3, 5, 9].

Edge unfoldings were first described in the early 16th century by Albrecht Dürer [16], implicitly raising the still-open question of whether every convex polyhedron has one (sometimes called Dürer’s conjecture). The question was first formally stated in 1975 by G. C. Shephard, although without reference to Dürer [17, 23]. It has been heavily studied since then, with progress of two types [15, 22]:

1. finding restricted classes of polyhedra, or generalized types of unfoldings, for which the existence of an unfolding can be guaranteed; and
2. finding generalized classes of polyhedra, or restricted types of unfoldings, for which counterexamples — *ununfoldable polyhedra* — can be shown to exist.

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Results guaranteeing the existence of an unfolding include:

- Every pyramid, prism, prismoid, and dome has an edge unfolding [15].
- Every sufficiently flat acutely triangulated convex terrain has an edge unfolding [21]. Consequentially, every acutely triangulated convex polyhedron can be unfolded into a number of planar pieces that is bounded in terms of the “acuteness gap” of the polyhedron, the minimum distance of its angles from a right angle.
- Every convex polyhedron has an affine transformation that admits an edge unfolding [18].
- Every convex polyhedron can be unfolded to a single planar piece by cuts interior to its faces [3, 14].
- Every polyhedron with axis-parallel sides can be unfolded after a linear number of axis-parallel cuts through its faces [10].
- Every triangulated surface (regardless of genus) has a “vertex unfolding”, a planar layout of triangles connected through their vertices that can be folded into the given surface [13].
- For ideal polyhedra in hyperbolic space, unlike Euclidean convex polyhedra or non-ideal hyperbolic polyhedra, every spanning tree forms the system of cuts of a convex unfolding into the hyperbolic plane.

Previous constructions of unfoldable polyhedra include the following results. A polyhedron is *topologically convex* if it is combinatorially equivalent to a convex polyhedron, meaning that its surface is a topological sphere and its graph is a 3-vertex-connected planar graph.

- Some orthogonal polyhedra and topologically convex orthogonal polyhedra have no edge unfolding, and it is NP-complete to determine whether an edge unfolding exists in this case [1, 8].
- There exists a convex-face star-shaped topologically convex polyhedron with no edge unfolding [20, 24].

- There exists a triangular-face topologically convex polyhedron with no edge unfolding [7].
- There exist edge-ununfoldable topologically convex polyhedra with as few as 7 vertices and 6 faces, or 6 vertices and 7 faces [4].
- There exists a topologically convex polyhedron that does not even have a vertex unfolding [2].
- There exist domes that have no *Hamiltonian unfolding*, in which the cuts form a Hamiltonian path through the graph of the polyhedron [12]. Similarly, there exist polycubes that have no Hamiltonian unfolding [11].
- There exists a convex polyhedron, equipped with 3-vertex-connected planar graph of geodesics partitioning the surface into regions metrically equivalent to convex polygons, that cannot be cut and unfolded along graph edges [6].

In this paper, we consider two questions left open by the previous work on edge-ununfoldable polyhedra with triangular faces [7], and strongly motivated by O'Rourke's recent results on unfoldings of acutely-triangulated polyhedra [21]. First, the previous counterexample of this type involved triangles with highly obtuse angles. Is this a necessary feature of the construction, or does there exist an unfoldable polyhedron with triangular faces that are all acute? Second, how far from being unfoldable can these examples be? Is it possible to cut the surfaces of these polyhedra into a bounded number of planar pieces (instead of a single piece) that can be folded and glued to form the polyhedral surface? (Both questions are motivated by previously posed analogous questions for convex polyhedra, as easier versions of Dürer's conjecture [15, Open Problems 22.12 and 22.17].)

We answer both of these questions negatively, by finding families of topologically convex edge-ununfoldable polyhedra with all faces acute triangles, in which any cutting of the surface into regions that can be unfolded to planar pieces must use an arbitrarily large number of pieces. Additionally, we use a similar construction to prove that there exist edge-ununfoldable *stacked polyhedra* [19], formed by gluing tetrahedra face-to-face with the gluing pattern of a tree, that also require an arbitrarily large number of pieces to unfold. We leave open the question of whether there exists an edge-ununfoldable stacked polyhedron with acute-triangle faces.

2 Hats

Our construction follows that of Bern et al. [7] in being based on certain triangulated topological disks, which

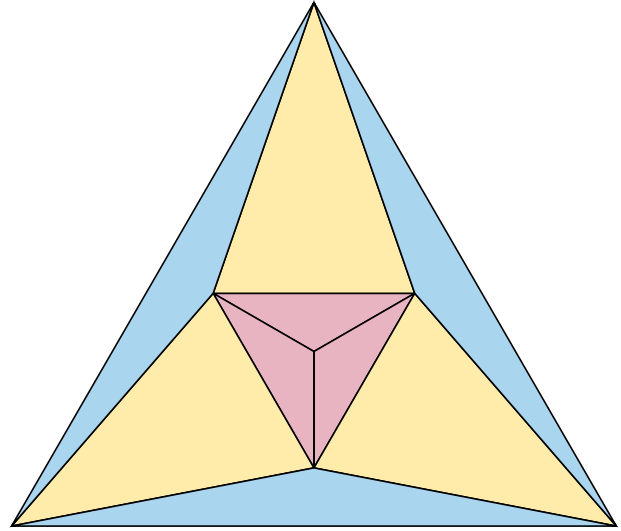


Figure 1: Combinatorial structure of a hat

they called *hats*. The combinatorial structure of a hat (in top view, but with different face angles than the hat we use in our proof) is shown in Figure 1: It consists of nine triangles, three of which (the *brim*, blue in the figure) have one edge on the outer boundary of the disk. The next three triangles, yellow in the figure, have a vertex but not an edge on the disk boundary; we call these the *band* of the hat. The central three triangles, pink in the figure, are disjoint from the boundary and meet at a central vertex; we call these the *crown* of the hat.

In both the construction of Bern et al. [7] and in our construction, the three vertices of the hat that are interior to the disk but not at the center all have negative curvature, meaning that the sum of the angles of the faces meeting at these vertices is greater than 2π . The center vertex, on the other hand, has positive curvature, a sum of angles less than 2π . When this happens, we can apply the following lemmas:

Lemma 2.1 *At any negatively-curved vertex of a polyhedron, any unfolding of the polyhedron that cuts only its edges and separates its surface into one or more simple polygons must cut at least two edges at each negatively-curved vertex.*

Proof. If only one edge were cut then the faces surrounding that vertex could not unfold into the plane without overlap. \square

Lemma 2.2 *Let D be a subset of the faces of a polyhedron, such that the polyhedron is topologically a sphere and D is topologically equivalent to a disk (such as a hat). Then in any unfolding of the polyhedron (possible cutting it into multiple pieces), either D is separated*

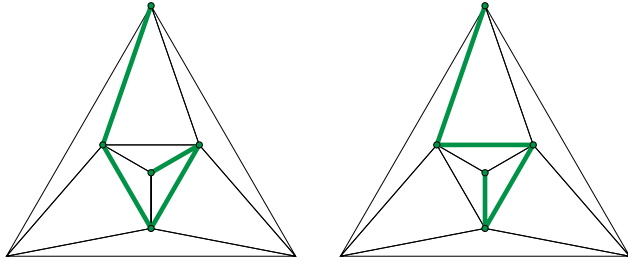


Figure 2: Two paths from a boundary vertex of a hat, through all three negatively curved vertices, to the center vertex

into multiple pieces by a path of cut edges from one boundary vertex of D to another or by a cycle of cut edges within D , or the set of cut edges within D forms a forest with at most one boundary vertex for each tree in the forest.

Proof. If the cut edges within D do not form a forest, they contain a cycle and the Jordan Curve Theorem implies that this cycle separates an interior part of the boundary from the exterior. If they form a forest in which some tree contains two boundary vertices, then they contain a boundary-to-boundary path within D , again separating D by the Jordan Curve Theorem. The only remaining possibility is a forest with at most one boundary vertex per tree. \square

Lemma 2.3 *For a hat combinatorially equivalent to the one in Figure 1, with positive curvature at the center vertex and negative curvature at the other three interior vertices, any unfolding that does not cut the hat into multiple pieces must cut a set of edges along a single path from a boundary vertex to the center vertex.*

Proof. By Lemma 2.2, each component of cut edges must form a tree with at most one boundary vertex within the hat. But every tree with one or more edges has at least two leaves, and every tree that is not a path has at least three leaves. By Lemma 2.1, the only non-boundary leaf can be the center vertex, so each component must be a path from the boundary to this vertex. \square

Up to symmetries of the hat, there are only two distinct shapes that the path of Lemma 2.3 from the boundary to the center of a hat can have (Figure 2). These two cuttings differ in how the crown triangles are attached to the band and to each other, but they both cut the brim and band triangles in the same way, into a strip of triangles connected edge-to-edge around the boundary of the hat.

Our key new construction is depicted in unfolded (but self-overlapping) form in Figure 3. It is a hat in which all triangles are acute and isosceles:

- The three brim triangles have apex angle 85° and base angle 47.5° .
- The three band triangles have base angle 85° and apex angle 10° .
- The three crown triangles are congruent to the band triangles, with base angle 85° and apex angle 10° .

As in the construction of Bern et al. [7], this leaves negative curvature (total angle 425° from five 85° angles) at the three non-central interior angles of the hat, and positive curvature (total angle 30°) at the center vertex, allowing the lemmas above to apply. The cut edges of the figure form a tree with a degree-three vertex at one of the negatively curved vertices of the hat, and a leaf at another negatively curved vertex, the one at which the self-overlap of the figure occurs. So the cutting in the figure does not match in detail either of the two path cuttings of Figure 2. Nevertheless, the brim and band triangles are unfolded as they would be for either of these two path cuttings. It is evident from the figure that this unfolding of the brim and band triangles cannot be extended to a one-piece unfolding of the entire hat: if a crown triangle is attached to the middle of the three unfolded band triangles (as it is in the figure) then there is no room on either side of it to attach the other two crown triangles, and a crown triangle attached to either of the other two band triangles would overlap the opposite band triangle. We prove this visual observation more formally below.

Lemma 2.4 *The hat with acute triangles described above has no single-piece unfolding.*

Proof. As we have already seen in Lemma 2.3, any unfolding (if it exists) must be along one of the two cut paths depicted in Figure 2. As a result, the unfolding of the brim and band triangles (but not the crown triangles) must be as depicted in Figure 3. In this unfolding, the three base sides of the unfolded band triangles form a polygonal chain whose interior angles (surrounding the central region of the figure where the pink crown triangles are attached) can be calculated as 105° .

A regular pentagon has interior angles of 108° , and has the property that each vertex lies on the perpendicular bisector of the opposite edge. Because the interior angles of the chain of base sides of band triangles are 105° , less than this 108° angle, it follows that the band triangle at one end of the chain extends across the perpendicular bisector of the base edge at the other end of the chain. Further, it does so at a point closer than the vertex of a regular pentagon sharing this same base edge (Figure 4).

If a crown triangle were attached to one of the two base edges at the ends of the chain of three base edges,

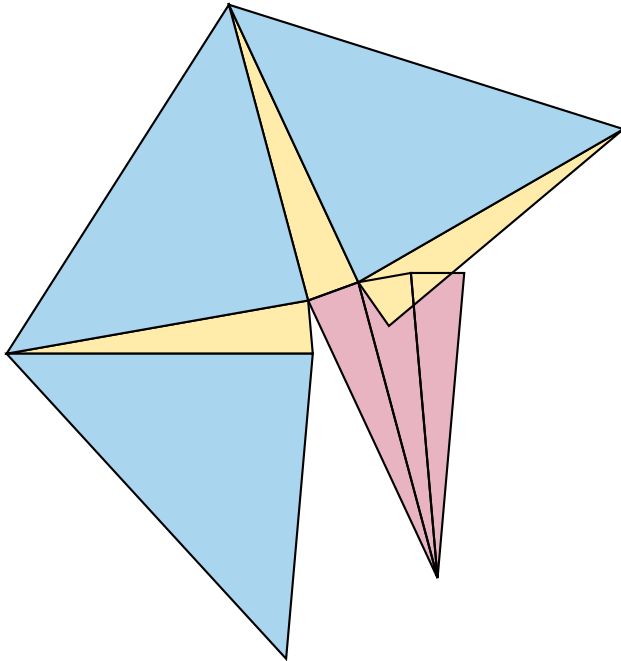


Figure 3: A hat made with acute isosceles triangles. Unlike Figure 2, the cuts made to form the self-overlapping unfolding shown do not form a path.

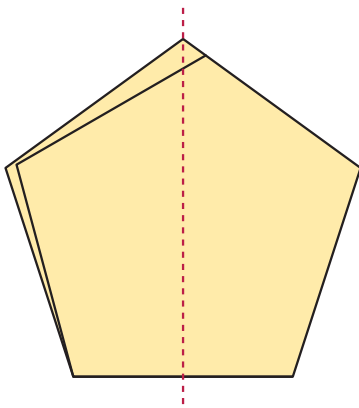


Figure 4: Each vertex of a regular pentagon lies on the perpendicular bisector of the opposite side; in a path of three equal edges with the tighter angle 105° , the last edge overlaps the perpendicular bisector of the first.

its altitude would lie along the perpendicular bisector of the base edge. And because the crown triangle has an apex angle of 10° , sharper than the angle of an isosceles triangle inscribed within a regular pentagon, its altitude extends across the perpendicular bisector farther than the regular pentagon vertex, causing it to overlap with the band triangle at the other end of the chain of three base edges.

Therefore, attaching a crown triangle to either the first or last of the band triangle base edges in the

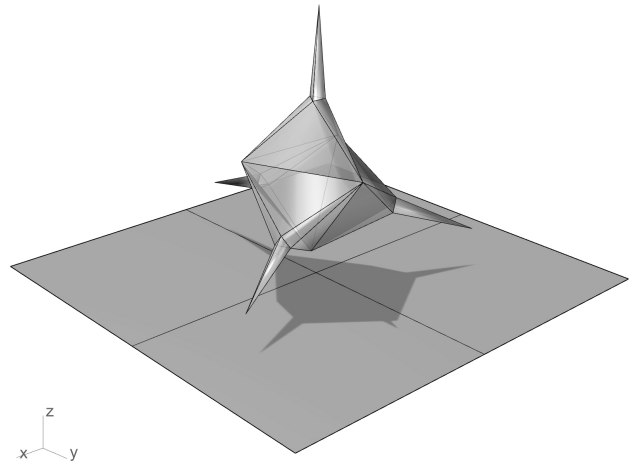


Figure 5: Tetrahedron with faces replaced by hats

chain of these three edges necessarily leads to a self-overlapping unfolding. However, these two ways of attaching a crown triangle are the only ones permitted by the two cases depicted in Figure 2. Attaching a crown triangle to the middle of the three base edges, as in Figure 3, can only be done by cutting along a tree that is not a path. Therefore, no unfolding exists. \square

The following construction is straightforward, and will allow us to construct polyhedra with multiple hats while keeping the hats disjoint from each other.

Lemma 2.5 *The hat with acute triangles described above can be realized in three-dimensional space, lying within a right equilateral-triangle prism whose base is the boundary of the hat.*

3 Acute Ununfoldable Polyhedra

We now use these hats to construct a topologically convex unfoldable polyhedron.

Theorem 3.1 *There exists a topologically convex unfoldable polyhedron whose faces are all acute isosceles triangles.*

Proof. Replace the four faces of a regular tetrahedron by acute-triangle hats, all pointing outward, as shown in Figure 5. Because each lies within a prism having the tetrahedron face as a base, they do not overlap each other in space. By Lemma 2.4, no hat can be unfolded into a single piece, so any possible unfolding (even one into multiple pieces) must cut each hat along some path between two of its three boundary vertices (at least; there may be more cuts besides these). The four paths

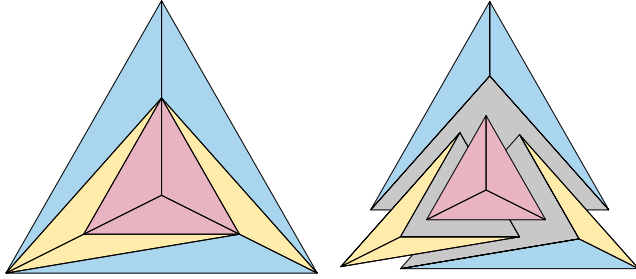


Figure 6: Hat for stacked polyhedra (top view, left, and exploded view as a stacked polyhedron, right)

formed in this way are disjoint except at their ends, and connect the four vertices of the tetrahedron, necessarily forming at least one cycle that separates the tetrahedron into at least two pieces. \square

Like the examples of Tarasov, Grünbaum, and Bern et al. [7, 20, 24], the resulting polyhedron is also star-shaped, with the center of the tetrahedron in its kernel.

4 Stacked Ununfoldable Polyhedra

A *stacked polyhedron* is a polyhedron that can be formed by repeatedly gluing a tetrahedron onto a single triangular face of a simpler stacked polyhedron, starting from a single tetrahedron [19]. To make unfoldable stacked polyhedra, we use a similar strategy to our construction of unfoldable polyhedra with acute-triangle-faces, in which we replace some faces of a convex polyhedron by hats. However, the acute-triangle hat that we used earlier cannot be used as part of a stacked polyhedron: in a stacked polyhedron, every non-face triangle is subdivided into three smaller triangles, but that is not true of the outer triangle of Figure 1. Instead, we use the hat shown in Figure 6. As before, it has three brim triangles, three band triangles, and three crown triangles, but they are arranged differently and less symmetrically. We make the brim and band triangles nearly coplanar, with shapes approximating those shown in the figure, but projecting slightly out of the figure so that the result can be constructed as a stacked polyhedron. We choose the crown triangles to be isosceles, and taller than the isosceles triangles inscribed in regular pentagons, as in our acute-triangle construction, so that (as viewed in Figure 6) they project out of the figure.

Lemma 4.1 *The hat described above has no single-piece unfolding.*

Proof. As with our other hat, the center vertex of this hat has positive curvature, and the other three interior vertices have negative curvature, so by Lemma 2.3 any unfolding of the hat that leaves it in one piece must form a path consisting of a single edge cutting from the

boundary to the crown, two edges cutting between the band and the crown, and one edge cutting to the center of the crown.

There are many more cases than there were in Figure 2, but we can avoid case-based reasoning by arguing that in each case, the brim and band triangles unfold in such a way that the three edges between the band and crown triangles form a polygonal chain with interior angles less than the 108° angles of the regular pentagon (in fact, close to 60° , because of the way we have constructed this part of the hat to differ only by a small amount from the top view shown in Figure 6). Therefore, just as in Figure 4, each edge at one end of this chain of three edges overlaps the perpendicular bisector of the edge at the other end of the chain.

Cutting along a path from a boundary edge of the hat to its center vertex forces the three crown triangles to be attached to the unfolded brim and band triangles on one of the two edges at the end of this path. However, our construction makes the three crown triangles tall enough to ensure that, no matter which of these two edges they are attached to, they will overlap the edge at the other end of the path at the point where it crosses the perpendicular bisector. \square

Theorem 4.2 *There exists an unfoldable stacked polyhedron.*

Proof. We replace the four faces of a regular tetrahedron with the hat described above. Each such replacement can be realized as a stacking of four tetrahedra onto the face, so the result is a stacked polyhedron. As in Theorem 3.1, each hat lies within a prism having the tetrahedron face as a base, so they do not overlap each other in space; and the set of edges cut in any unfolding must include at least four paths between the four tetrahedron vertices, necessarily forming a cycle that cuts one part of the polyhedron surface from the rest. \square

A stacked hat with the same combinatorial structure as the one used in this construction, with the center vertex positively curved and the surrounding three vertices negatively curved, cannot be formed from acute triangles, because that would leave the degree-four vertex with positive curvature. We leave as an open question whether it is possible for an unfoldable stacked polyhedron to have all faces acute.

5 Very Unfoldable Polyhedra

Both families of examples above can be made into very unfoldable families. In both cases, the approach is the same: instead of starting from a tetrahedron, we start from a polyhedron with many triangular faces, and show that attaching hats to more and more triangles requires more and more unfolded pieces.

Theorem 5.1 *There exist topologically convex polyhedra with acute isosceles triangle faces such that any unfolding formed by cutting along edges into multiple non-self-overlapping pieces requires an unbounded number of pieces.*

Proof. For any integer $k \geq 1$, refine the regular tetrahedron by subdividing each edge into k equal-length edges and subdivide each face into a regular grid of $\sum_{i=1}^k (2i - 1) = k^2$ equilateral triangles of side length $1/k$, for a total of $4k^2$ faces and (by inclusion-exclusion) $\sum_{i=1}^{k+1} i - 6(k + 1) + 4 = 2k^2 + 2$ vertices. Replace each equilateral triangular face by an acute-triangle hat pointing outward. As in Theorem 3.1, each hat lies within a prism having the face of the tetrahedron as a base, so they do not overlap each other in space; and any unfolding into multiple pieces must, in each hat, either cut along a cycle within the hat or cut along some path connecting two of its three boundary vertices. Let c be the number of cycles within hats cut in this way, so that we have a system of at least $4k^2 + c$ disjoint paths connecting pairs of subdivided-tetrahedron vertices.

Now consider cutting the polyhedron surface along these paths, one by one. Each cut either connects two subdivided-tetrahedron vertices that were not previously connected along the system of cuts, or two subdivided-tetrahedron vertices that were previously connected. If cutting along a path connects two vertices that were not previously connected, it reduces the number of connected components among these vertices; this case can happen at most $2k^2 + 1$ times. If cutting along a path connects two vertices that were previously connected, then that path and the path through which they were previously connected form a Jordan curve that separates off two parts of the surface from each other. Because there are $4k^2 - c$ paths connecting pairs of subdivided-tetrahedron vertices, only $2k^2 + 1$ of which can form new connections, this case must happen at least $2k^2 - 1 - c$ times. Because the surface started with a single piece and undergoes at least $2k^2 - 1 - c$ separations, it ends up with at least $2k^2 - c$ pieces, which together with the c additional pieces formed by cycles within hats, form a total of at least $2k^2$ pieces. \square

Theorem 5.2 *There exist topologically convex stacked polyhedra such that any unfolding formed by cutting along edges into multiple non-self-overlapping pieces requires an unbounded number of pieces.*

Proof. For any integer $k \geq 0$, refine the regular tetrahedron by choosing any face and attaching to the face a very shallow tetrahedron whose apex is near the in-center of the face, effectively splitting the face into three faces, and repeating this process a total of k times. Because each attachment increases the number of faces by 2 and the number of vertices by 1, the result is a stacked

polyhedron with $4 + 2k$ triangular faces (not necessarily equilateral) and $4 + k$ vertices. Replace each triangle with a version of the hat from Section 4 pointed outward, using the availability flexibility to make the interface between the band and crown an equilateral triangle near the in-center of the original triangle. As in Theorem 4.2, the result is a stacked polyhedron; each hat lies within a prism having the face of the tetrahedron as a base, so they do not overlap each other in space; and any unfolding into multiple pieces must cut each hat along some path connecting two of its three boundary vertices. As in Theorem 5.1, at most $3 + k$ such paths can decrease the number of connected components among the $4 + k$ vertices, leaving at least $1 + k$ paths that separate the surface into at least $2 + k$ pieces. \square

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