Computational Complexity of Piano-Hinged Dissections

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SUMMARY We prove NP-completeness of deciding whether a given loop of colored right isosceles triangles, hinged together at edges, can be folded into a specified rectangular three-color pattern. By contrast, the same problem becomes polynomially solvable with one color or when the target shape is a tree-shaped polyomino.
key words: GeoLoop, hinged dissection, Ivan's Hinge, NP-hardness, paper folding

1. Introduction

One of the simplest and most practical physical folding structures is that of a *hinge*, as in most doors or attaching the lid to a grand piano. Frederickson [1] introduced a way to make folding structures out of such hinges that can change their shape between "nearly 2D" shapes. The basic idea is to thicken a (doubly covered) 2D polygon by extruding it orthogonally into a height-2ε 3D prism, divide that prism into two height-ε layers, further divide those layers into ε-thickened polygonal pieces, and hinge the pieces together with hinges along shared edges. The goal in a *piano-hinged dissection* is to find a connected hinging of ε-thickened polygonal pieces that can fold into two (or more) different 2ε-thickened polygons.

Piano-hinged dissections are meant to be a more practical form of *hinged dissections*, which typically use point hinges and thus are more difficult to build [1]. Although hinged dissections have recently been shown to exist for any finite set of polygons of equal area [2], no such result is known for piano-hinged dissections.

Here we study a family of simple piano-hinged dissections, which we call a *piano-hinged loop*: $4n$ identical ε-thickened right isosceles triangles, alternating in orientation, and connected into a loop by hinges on the bottoms of

their isosceles sides; see Fig. 1. Frederickson [1, Chapter 11] mentions without proof that this piano-hinged dissection can fold into any (2ε-thickened) *n*-omino, that is, any connected edge-to-edge joining of *n* unit squares.

Three commercial puzzles, shown in Fig. 2, consist of piano-hinged loops. *GeoLoop* is a piano-hinged loop with $n = 6$ that was patented by Kenneth Stevens in 1993/1994 [3] and sold by Binary Arts* in 1996. The pieces alternate between two colors, and by a checkerboard property of the piano-hinged loop, the resulting squares of any polyomino will alternate in color (on either side), so this puzzle is effectively *uncolored*. *Ivan's Hinge* is a piano-hinged loop with $n = 4$ that was patented by Jan Essebaggers and Ivan Moscovich in 1993/1994 [4] and sold by Paradigm Games in the mid-to-late 1990s [1] and recently by Fat Brain Toys**. Each piece is colored irregularly with one of two colors, and the goal in this puzzle is to make not only the specified tetromino shape but also the specified two-color pattern. *Tony's Hinge* is a variation of Ivan's Hinge, sold by Kellogg Company in 1988 but also copyright by Ivan Moscovitch and made by Paradigm Games. It uses colored images and requires putting certain images in particular places, in addition to the color constraints.

Our results.

In this paper, we investigate the computational complexity of folding colored and uncolored piano-hinged loop puzzles into *n*-ominoes.

First we consider the uncolored piano-hinged loop, as in GeoLoop. For completeness, we prove Frederickson's claim that this loop can realize any 2ε-thickened *n*-omino, by mimicking a simple inductive argument for hinged dissections of polyominoes from [5]. For the special case of *tree-shaped* polyominoes, where the dual graph of edge-to-edge adjacencies among unit squares forms a tree, we prove further that the folding of the piano-hinged loop is unique up to cyclic shifts of the pieces in the loop.

Next we consider colored piano-hinged loops, as in Ivan's Hinge. For tree-shaped polyominoes, the previous uniqueness result implies that the problem can be solved in $O(n^2)$ time by trying all cyclic shifts. (In particular, this observation makes the $n = 4$ case of Ivan's Hinge easy to solve in practice, as each tetromino has either 1 or 4 span-
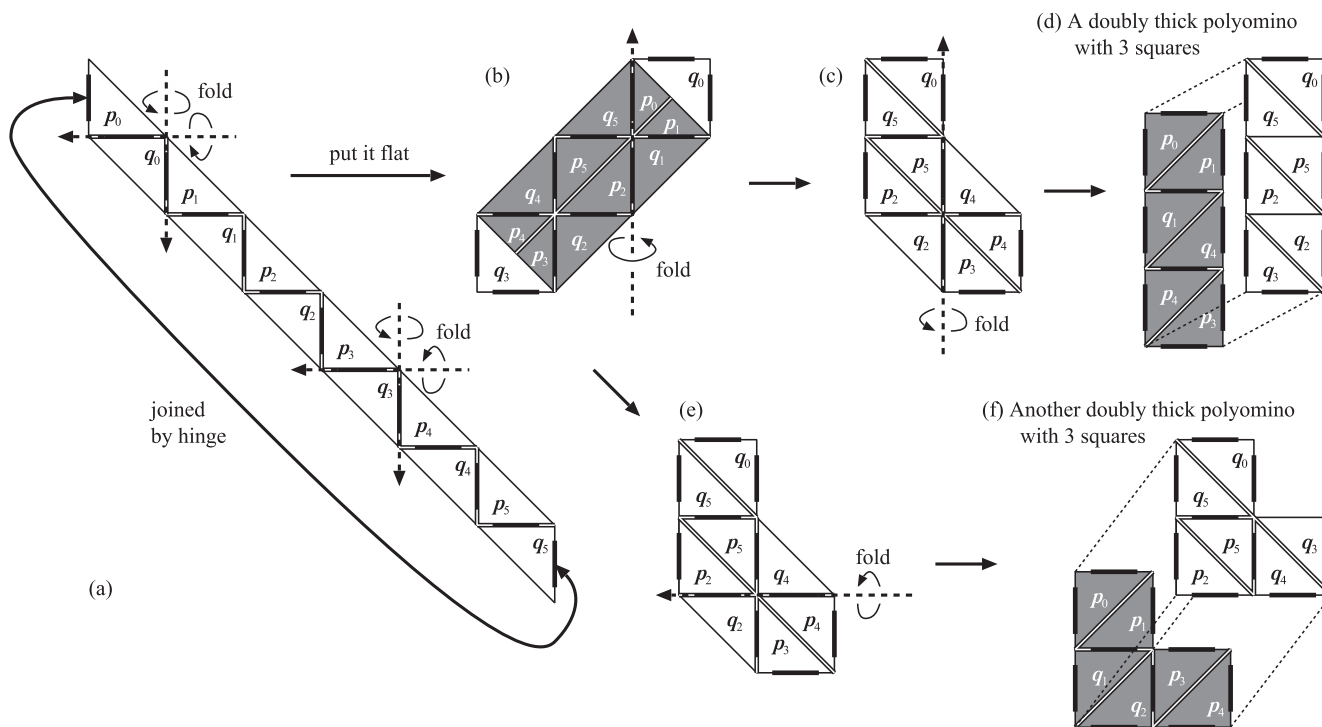
Fig. 1   Small piano-hinged loop and its folding way.

Fig. 2   GeoLoop, Ivan's Hinge, and Tony's Hinge.

ning trees to try.) For general polyominoes, we prove that the problem is NP-complete even if the number of colors is 3, each piece is colored uniformly one color, and the target shape is a rectangle.

2.   Preliminaries

A *piano-hinged loop* consists of a loop of $4n$ consecutive isosceles right triangles $p_0, q_0, p_1, q_1, \ldots, p_{2n-1}, q_{2n-1}$, as shown in Fig. 1. Every two consecutive triangular pieces share one of two isosceles edges. The p_i's have a common orientation (collinear hypotenuses when unfolded), as do the q_i's, and these two orientations differ from each other. Each shared edge is a *piano hinge* on the back side that permits bending inward (bringing the two back sides together).

In a *folded state* of the piano-hinged loop into a doubly covered polyomino, (1) each piano hinge is flat (180°) or folded inward (360°); and (2) each unit square of the poly-

omino consumes exactly four triangles, with two triangles on the front and two on the back side. Thus, in any folded state, the exposed surface consists of all front sides of the pieces, while the back sides of all pieces remain hidden on the inside. In other words, each hinge is folded into the same direction, or stays flat. Therefore, we can ignore the color of the back side of each piece, so for simplicity we can assume that each piece has a uniform color (instead of a different color on each side). Let $c(p_i)$ and $c(q_i)$ denote the color of piece p_i and q_i.

For the resultant polyomino P of n unit squares, we define the *connection graph* $G(P) = (V, E)$ as follows: V consists of n unit squares, and E contains an edge $\{u, v\}$ if and only if squares u and v are adjacent in P. Having $\{u, v\} \in E$ is a necessary but not sufficient condition for there to be a hinge connecting the four pieces representing square u to the four pieces representing square v; if there is such a hinge, we call u and v *joined*. We note that P is doubly covered

polyomino; that is, u and v may have two possible hinges on their front side and back side. We say u and v are joined when they are joined by at least one of these possible hinges.

The *uncolored piano-hinged loop problem* asks whether a given polyomino can be constructed as (the silhouette of) a folded state of a given piano-hinged loop. The "silhouette" phrasing allows the folding to have unjoined squares, which are adjacent in the polyomino but not attached by a hinge in the folded state. The *colored piano-hinged loop problem* asks whether a given colored polyomino pattern can be similarly constructed from a given colored piano-hinged loop.

The piano-hinged loop has a simple checkerboarding property as seen in GeoLoop in Fig. 2:

Observation 1: Consider two adjacent squares u and v (in any direction) in a polyomino P, obtained as a folded state of a piano-hinged loop. Without loss of generality, assume that the top side of u contains (the front side of) triangle p_i. Then (1) the other triangle of u on front side is p_j for some j, (2) the backside of u contains two qs, (3) the front side of v contains two qs, and (4) the backside of v contains two ps.

Proof: In a polyomino, each piano-hinge takes one of two possible state; it is folded or flat. Let t be any triangle, and t' be the next triangle sharing a hinge (in any direction). Without loss of generality, t is p_i on the front side of a square u. Then t' should be $q_{i+1 \bmod (2n)}$. When the hinge is flat, t' is on the front side of the neighbor square v of the square u containing t. If the hinge is folded, t' is on the back side of the square u containing t. These two cases imply the checkerboarding property stated in the observation. That is, if we color ps in white and qs in black, we always have a checkerboarding color pattern regardless of the shape of the resulting polyomino as seen in GeoLoop in Fig. 2. □

Ivan's Hinge has a group of triangles that are monochromatic as assumed above, and a group of triangles with different colors on their front and back sides. However, these groups directly correspond to the parity classes in Observation 1. Hence, for each unit square, the front side consists of two triangles from the same group, and the back side consists of two triangles from the other group. Thus, from a theoretical point of view, we can again effectively assume that the pieces are monochromatic. (Practically, the differing colors can vary the color patterns, which can help visually.)

3. Uncolored Piano-Hinged Loop

We begin with the universality theorem of GeoLoop, claimed by Frederickson [1]:

Theorem 2 ([1]): Any polyomino P of n unit squares can be realized as a folded state of the piano-hinge loop of $4n$ pieces.

Proof: It is easy to see that for the case $n = 1$: The flat state is already a unit square. In fact, Fig. 1(d) and Fig. 1(f) explicitly show the case $n = 3$. We use an induction for
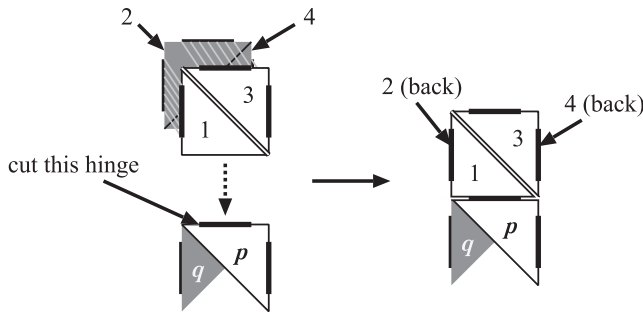

Fig. 3 An extension of four triangles.

n. Suppose all polyomino of k unit squares can be folded from the piano-hinge loop of $4k$ pieces. Let P be a polyomino of $k + 1$ unit squares. Consider a spanning tree T of the connection graph $G(P)$. Let v be a leaf of T, and u the parent of v in T. We remove the corresponding square v from P and obtain a smaller polyomino P' of k unit squares. By inductive hypothesis, P' can be realized as a folded state of the piano-hinge loop of $4k$ pieces. Since P' is obtained from P by removing v from u, the corresponding square u in P' contains a boundary of P' such that this edge of u was attached to v in P. It is not difficult to see that each boundary of a polygon from a piano-hinge loop should be a hinge shared by two consecutive triangles p_i and q_i, or q_i and p_{i+1} for some i. So we cut this boundary of u, attach four consecutive isosceles right triangles (as illustrated in Fig. 3; in the figure, the boundary of two consecutive triangles p and q is cut, and a new four triangles 1, 2, 3, 4 are attached there, which yields the extended sequence $p, 1, 2, 3, 4, q$), and fold them to form v in P. □

Once we fix the spanning tree T of $G(P)$, we claim that the folded state is uniquely determined up to cyclic shift of the pieces. Both this corollary and the previous theorem follow from a simple argument of repeatedly pruning leaves in the graph of joinings.

Corollary 3: Let P be any polyomino of n unit squares such that $G(P)$ is a tree. Then it can be uniquely folded from the piano-hinge loop of $4n$ pieces, up to cyclic shift of the pieces.

For a given tree-shaped polyomino, the piano-hinge loop traverses the tree in the same manner as the depth-first search without crossing. That is, if we imagine that we are in the maze in the form of the tree, and traverse the maze by the right-hand rule, then we traverse each edge twice, and this is the order followed by the piano-hinge loop. This intuition will be useful in some proofs in this paper.

4. General Piano-Hinged Loop

Consider a polyomino P in which pieces p_i and q_i have colors $c(p_i)$ and $c(q_i)$, respectively. When the connection graph $G(P)$ is a tree (or the spanning tree of $G(P)$ is explicitly given), we still have a polynomial time algorithm to solve the problem:
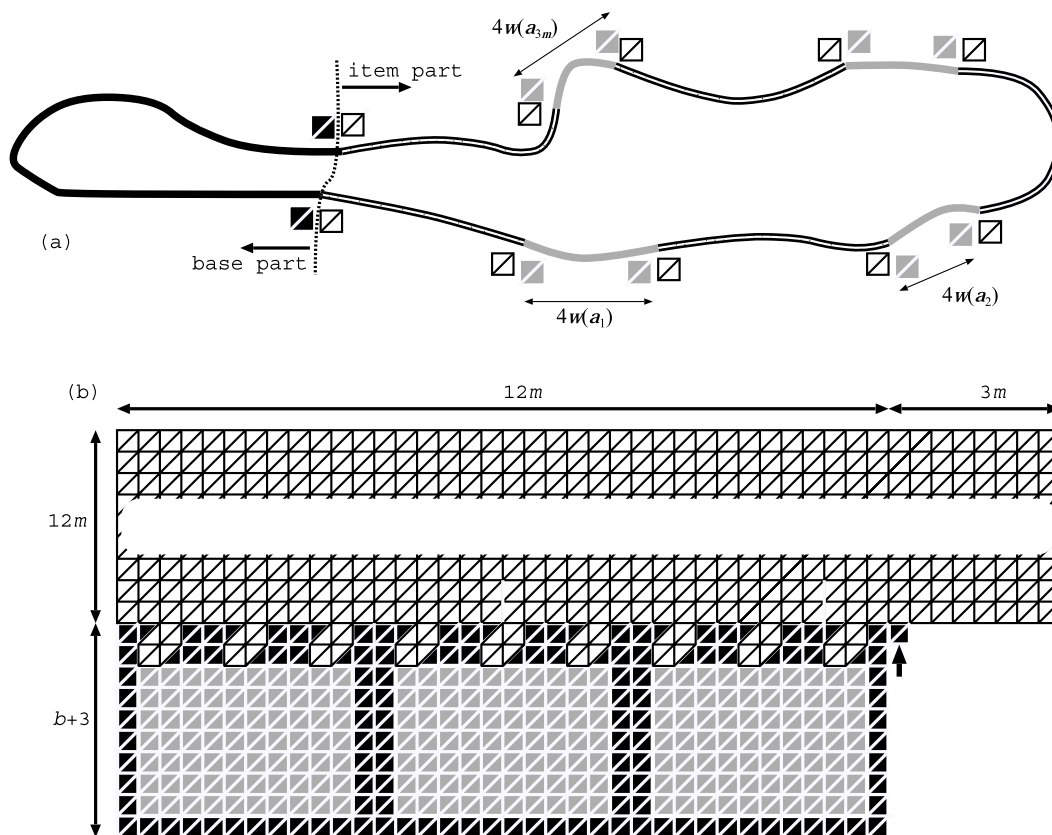
Fig. 4 Outline of the construction.

Theorem 4: Let P be any polyomino of n unit squares such that $G(P)$ is a tree T. Then the general piano-hinge loop problem can be solved in $O(n^2)$ time.

Proof: Once we fix a position of one triangle in the loop on T, the folded state forming T is uniquely determined by Corollary 3. Hence, for each triangle in the loop, we temporarily put it on a fixed point on T, and check if the color pattern is achieved in this case in linear time. This gives us an $O(n^2)$ time algorithm. □

Next we turn to the case that P is a general polyomino, where the problem is NP-complete.

Theorem 5: The colored piano-hinge loop problem is NP-complete, even if the number of colors is 3 and the target polyomino is a rectangle.

Proof: It is clear that this problem is in NP, we in the following show the hardness by reducing 3-PARTITION, defined as follows.

3-PARTITION (cf. [6])
INSTANCE: A finite set $A = \{a_1, a_2, \ldots, a_{3m}\}$ of $3m$ weighted elements with $w(a_j) \in \mathbb{Z}_+$, where $w(a_j)$ gives the weight of a_j, and a bound $B \in \mathbb{Z}_+$ such that each a_j satisfies $B/4 < w(a_j) < B/2$ and $\sum_{j=1}^{3m} w(a_j) = mB$.
QUESTION: Can A be partitioned into m disjoint sets $A^{(1)}, A^{(2)}, \ldots, A^{(m)}$ such that $\sum_{a_j \in A^{(i)}} w(a_j) = B$ for $1 \leq i \leq m$?

It is well-known that 3-PARTITION is strongly NP-complete, meaning that it is NP-hard even if the input is written in unary notation [6]. In the following, we assume that $B = 10b$ for some positive integer b (otherwise, multiply all values by 10). If A has a solution, we can observe that each $A^{(i)}$ contains exactly three items since $B/4 < w(a_j) < B/2$ for each i and j.

The outline of the construction is illustrated in Fig. 4. Our piano-hinge loop L consists of two parts (Fig. 4(a)). The first part, which is called *base part*, is a series of black triangles that will form m empty *bins* such that each bin will filled by B gray unit squares. The second part, which is called *item part*, is a series of alternating gray and white triangles. The ith consecutive gray triangles represents the weight of an element a_i for each i, and consecutive white triangles will be used to rearrange the items to put them into bins in an arbitrary way.

Before precise construction, we here introduce a *crossover gadget*, which is usually complicated and important part for such a reduction. In the piano-hinge loop, it is easy to cross since each square is doubly covered. It is achieved as is illustrated in Fig. 5. In the figure, it is depicted how a white piano-hinge loop of 16 pieces and a light gray piano-hinge loop of 16 pieces cross each other. Each dark gray area is the back side of each colored triangle. From the initial position (a), we fold as in the figure. Then we have two squares of area 2 in two loops such that they are not
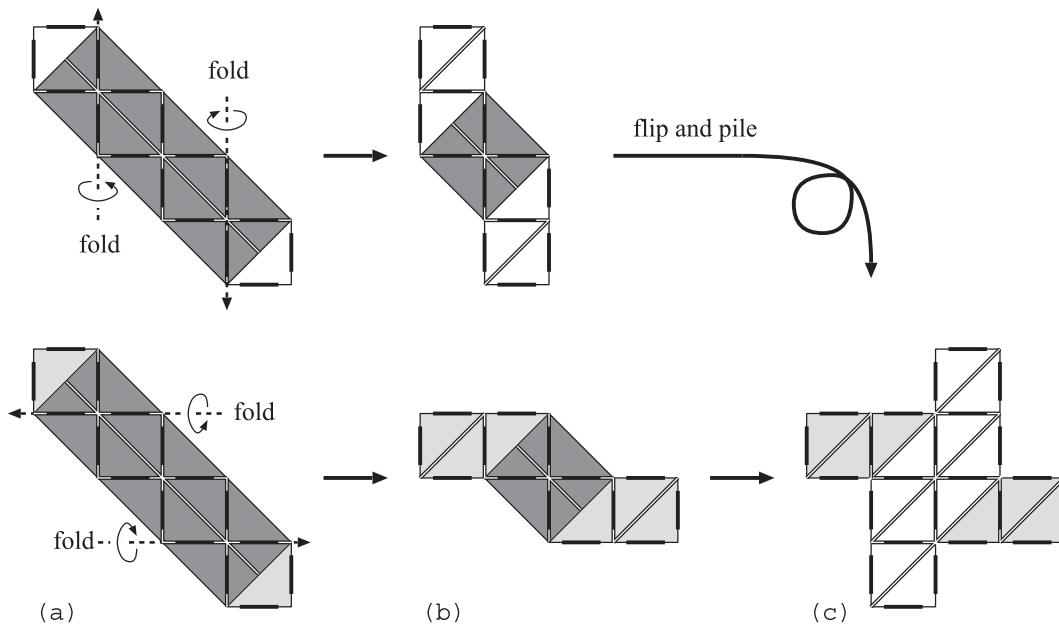
Fig. 5 Crossover gadget.

covered properly as in (b). Hence we flip the white loop vertically, and put it on the light gray loop so that two squares doubly cover one square as in (c). We note that the resultant polyomino of 8 unit squares in (c) is doubly covered one. Using this crossover gadget, we can make a crossing of a long vertical sequence of unit squares and a long horizontal sequence of unit squares. We note that these two sequences are shifted one unit when they cross. The fixing of these shifts is not required in the construction. In fact, all crossings occur in the crossing phase in Fig. 6(c), but the gaps by the shifts are filled in the next filling phase in Fig. 6(d).

Now we turn to the precise construction of the general piano-hinge loop of $4(12m(12m + b + 3) + 36m^2 + 1)$ pieces from an instance of 3-PARTITION $A = \{a_1, a_2, \ldots, a_{3m}\}$ and $B = 10b$. The base part is simple. In Fig. 4(b), the black triangles doubly cover the black area, and four of each back side of white slanted hexagons are covered by black triangles as in Fig. 5(c). Thus, in total, the number BL of black triangles is equal to $4(12m \times 3 - 3m \times 8 + b \times 2m + 1) = 32m + 8mb + 4$.

The upper rectangle consists of $15m \times 12m$ unit squares. This rectangle is divided into two rectangles of size $12m \times 12m$ and $3m \times 12m$. Roughly, each of $a_1, a_2, \ldots$ will consume a rectangle of size $12m \times 4$ from bottom to top in the left rectangle, and they are connected in the right rectangle of size $3m \times 12m$. At the right rectangle, each a_i uses $8i - 4$ unit squares. Let the item part consist of W_0 white triangles, G_1 gray triangles, W_1 white triangles, ..., G_{3m} gray triangles, and W_{3m} white triangles in this order. For each $i = 1, 2, \ldots, 3m$, we set $G_i = 4w(a_i)$. We also set $W_0 = 4(2 \times 12m + 4) + 4$, $W_i = 4(4 \times 12m + (8(i + 1) - 4)) + 8$ with $0 < i < 3m$, and $W_{3m} = 4(2 \times 12m + 24m - 4) + 4$. Intuitively, W_0 consists of the triangles in two lines before a_1 $(= 4(2 \times 12m))$, in right rectangle $(= 4 \times 4)$, and the triangles

just before a_1 $(= 4)$. W_i consists of the triangles in two lines after a_i $(= 4(2 \times 12m))$, two lines before a_{i+1} $(= 4(2 \times 12m))$, in right rectangle $(= 4 \times (8i - 4))$, the triangles just after a_i $(= 4)$, and before a_{i+1} $(= 4)$. The last W_{3m} consists of the triangles in two lines after a_{3m} $(= 4(2 \times 12m))$, in right rectangle $(= 4 \times (24m - 4))$, the triangles just after a_{3m} $(= 4)$.

The pattern that the loop should represent is illustrated in Fig. 4(b): The upper half is a rectangle of size $15m \times 12m$ that consists of all white triangles. The lower half is a rectangle of size $12m \times (b + 3)$ with one black square on the top right side, which is indicated by an arrow in the figure. The left rectangle is almost surrounded by black squares, and this area consists of m rectangular bins of size $10 \times b$. Each bin is filled by gray squares, and it is connected to the upper white area by three hexagonal white triangles as shown in the figure.

This is end of the construction of the general piano-hinge loop with its required pattern. It is easy to see that the reduction can be done in polynomial time. Hereafter, we sometimes abuse the notation B, W_i, G_i as the set of triangles in the sequence.

We first observe the gray squares in each G_i. It is not difficult to see that any crossing yields a unit square that consists of two triangles coming from the crossing sequences. However, all gray triangles are in monochromatic squares. Thus, there is no gray sequence crossing cross the different color sequences. Thus, if the loop makes the pattern, all triangles in G_i appear in the same bin. That is, if the desired pattern appears, we obtain a certain partition of A, and each gray sequence enters and exits at a white gate on the top of a bin. By the universality theorem 2, once we have the partition of A, we always can fill the bin. More precisely, when a bin can filled by gray triangles from G_i, G_j, and G_k, we first divide the area of the bin into three subareas such that
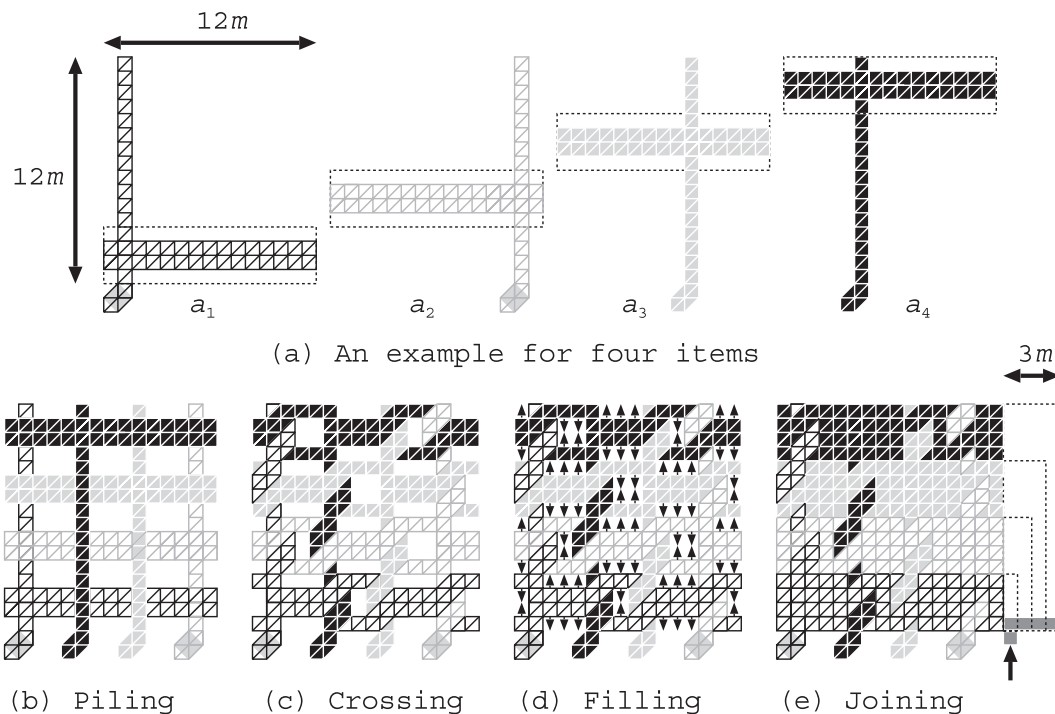
(a) An example for four items

(b) Piling   (c) Crossing   (d) Filling   (e) Joining

Fig. 6 The placement of white triangles.

each subarea is attached to one of three white gate, and each subarea consists of G_i, G_j, or G_k triangles. Then, the universality theorem guarantees that we can doubly cover each subarea by corresponding gray triangles.

By above observations, it is clear that if the general piano-hinged loop has a solution, so does the 3-PARTITION. Therefore, we now show that if the 3-PARTITION has a solution, the general piano-hinged loop has a solution.

We first consider the black triangles. Each hexagonal white triangles in the pattern represents the crossing of white vertical sequence and black horizontal sequence. That is, each hexagonal white triangles has four black triangles in its back side. Thus, in the lower rectangle, all black triangles are connected. On the other hand, using the universality theorem, the black triangles to form m bins illustrated in Fig. 4(b), with two endpoints comes to the upper right square (indicated by an arrow in the figure). The packing of the gray squares has been already discussed above. Thus, it is sufficient to show how can we arrange the gray triangles to arbitrary gate of a bin using white triangles W_0, W_1, ..., W_{3m} in the upper white rectangle.

We here show the way of an arrangement of the white triangles for any given ordering of $a_1, a_2, \ldots, a_{3m}$. First we pay attention to the left rectangle of size $12m \times 12m$. For each a_i, we design a cross of a $1 \times 12m$ rectangle and $12m \times 2$ rectangle as in Fig. 6(a). We note that, in the figure, the colors are used to distinguish the gadgets, but all triangles are white in the loop. The heights of $12m \times 2$ rectangles represent the indices of a_i. (In the figure, a_1, a_2, a_3, a_4 is arranged from left to right.) The placement of $1 \times 12m$ rect-

angle represents the position of bin. (In the figure, the items are put in bins in order of a_1, a_4, a_3, a_2.) Now, we pile them in the same area (Fig. 6(b)), and apply the crossing gadget at each crossing point (Fig. 6(c)). Now, most unit squares in $12m \times 2$ rectangle have an empty neighbor. This neighborhood is checked in each dotted rectangle in Fig. 6(a), and then each empty area has exactly one corresponding neighbor as shown in Fig. 6(d). So we fill the empty area by the corresponding neighbor in the same way in Fig. 3. After the process, we can check that each number of squares used for each a_i is the same.

Lastly, we join all gadgets above with one black square (indicated by an arrow in Fig. 4(b)) in the left rectangle of size $3m \times 12m$, which can be done in a straightforward way as shown in Fig. 6(e).

In the final step, we show that any cross of a $1 \times 12m$ rectangle and $12m \times 2$ rectangle in Fig. 6(a) is constructible. We here give a sketch of the adjustment of the length of the cross since the details are so fine that makes the argument unclear. For each i, we first wrap up the bottom half of the horizontal bar, go down, put the gray triangles in a bin. To do that, we use $4(2 \times 12m)$ triangles form W_{i-1}. Then, we go up with wrapping up the horizontal bars and vertical bars to the top consuming $4(2 \times 12m)$ triangles from W_i. To make a cross in Fig. 6(a), we need only $4(3 \times 12m)$ triangles. The other $4 \times 12m$ triangles are used to fill up the gaps in Fig. 6(d). In Fig. 6(d), we drew that each upper square in the horizontal bar covers the upper gap, and each lower square covers the lower gap to make the idea clear. However, by Theorem 2, we can arrange that any square can cover them. Therefore, using the squares before/after gray triangles ap-

propriately, we can adjust the crossing point. It is not difficult to see that the extra $4 \times 12m$ triangles are enough to put the crossing point arbitrary. Therefore, we can put the gray triangles into any bin. □

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