(Non)existence of Pleated Folds: How Paper Folds Between Creases[*]

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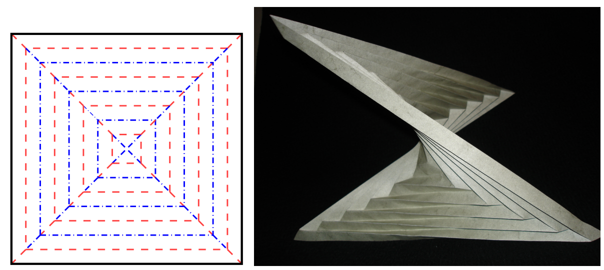
Abstract. We prove that the pleated hyperbolic paraboloid, a familiar origami model known since 1927, in fact cannot be folded with the standard crease pattern in the standard mathematical model of zero-thickness paper. In contrast, we show that the model can be folded with additional creases, suggesting that real paper "folds" into this model via small such creases. We conjecture that the circular version of this model, consisting of concentric circular creases, also folds without extra creases.

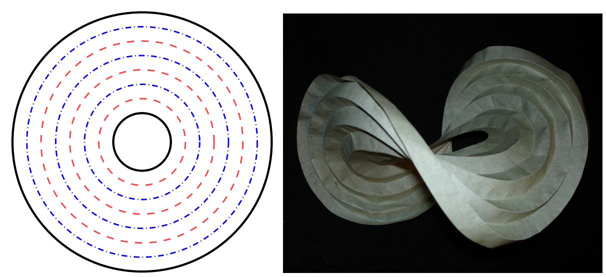
At the heart of our results is a new structural theorem characterizing uncreased intrinsically flat surfaces—the portions of paper between the creases. Differential geometry has much to say about the local behavior of such surfaces when they are sufficiently smooth, e.g., that they are torsal ruled. But this classic result is simply false in the context of the whole surface. Our structural characterization tells the whole story, and even applies to surfaces with discontinuities in the second derivative. We use our theorem to prove fundamental properties about how paper folds, for example, that straight creases on the piece of paper must remain piecewise-straight by folding.

1. Introduction. A fascinating family of *pleated* origami models use extremely simple crease patterns—repeated concentric shapes, alternating mountain and valley—yet automatically fold into interesting 3D shapes. The most well-known is the *pleated hyperbolic paraboloid*, where the crease pattern is concentric squares and their diagonals. As the name suggests, it has long been conjectured, but never formally established, that this model approximates a hyperbolic paraboloid. More impressive (but somewhat harder to fold) is the *circular pleat*, where the crease pattern is simply concentric circles, with a circular hole cut out of the center. Both of these models date back to the Bauhaus, from a preliminary course in paper study taught by Josef Albers in 1927–1928 [Win69, p. 434], and taught again later at Black Mountain College in 1937–1938 [Adl04, pp. 33, 73]; see [DD08]. These models owe their popularity today to origamist Thoki Yenn, who started distributing the model sometime before 1989. Examples of their use and extension for algorithmic sculpture include [DDL99, KDD08].



(a) Standard mountain-valley pattern. (b) Photograph of physical model. [Jenna Fizel]

Figure 1: Pleated hyperbolic paraboloid.



(a) Mountain-valley pattern. (b) Photograph of physical model. [Jenna Fizel]

Figure 2: Circular pleat.

The magic of these models is that most of the actual folding happens by the physics of paper itself; the origamist simply puts all the creases in and lets go. Paper is normally elastic: try wrapping a paper sheet around a cylinder, and then letting go—it returns to its original state. But *creases* plastically deform the paper beyond its yield point, effectively resetting the elastic memory of paper to a nonzero angle. Try creasing a paper sheet and then letting go—it stays folded at the crease. The harder you press the crease, the larger the desired fold angle. What happens in the pleated origami models is that the paper

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tries to stay flat in the uncreased portions, while trying to stay folded at the creases, and physics computes a configuration that balances these forces in equilibrium (with locally minimum free energy).

But some mathematical origamists have wondered over the years [Wer05]: do these models actually *exist*? Is it really possible to fold a piece of paper along exactly the creases in the crease pattern of Figures 1 and 2? The first two authors have always suspected that both models existed, or at least that one existed if and only if the other did. But we were wrong.

2. Our Results. We prove that the hyperbolic-paraboloid crease pattern of Figure 1(a) does not fold using exactly the given creases. In proving this impossibility, we develop a structural characterization of how uncreased paper can fold (hence the title of this paper). Surprisingly, such a characterization has not been obtained before. An intuitive understanding (often misquoted) is that paper folds like a ruled surface, but that claim is only true locally (infinitesimally) about every point. When the paper is not smooth or has zero principal curvature at some points, the truth gets even subtler. We correct both of these misunderstandings by handling nonsmooth (but uncreased) surfaces, and by stating a local structure theorem flexible enough to handle zero curvatures and all other edge cases of uncreased surfaces.

In contrast, we conjecture that the circular-pleat crease pattern of Figure 2(a) folds using exactly the given creases, when there is a hole cut out of the center. A proof of this would be the first proof of existence of a curved-crease origami model (with more than one crease) of which we are aware. Existing work characterizes the local folding behavior in a narrow strip around a curved crease, and the challenge is to extend this local study to a globally consistent folding of the entire crease pattern.

Another natural remaining question is what actually happens to a real pleated hyperbolic paraboloid like Figure 1(b). One conjecture is that the paper uses extra creases (discontinuities in the first derivative), possibly many very small ones. We prove that, indeed, simply triangulating the crease pattern and replacing the four central triangles with just two triangles, results in foldable crease pattern, shown in Figure 3, Our proof of this result is quite different in character, in that it is purely computational instead of analytical. We use interval arithmetic to establish with certainty that the exact object exists for many parameter values, and its coordinates could even be expressed by radical expressions in principle, but we are able only to compute arbitrarily close approximations, shown in Figure 4.

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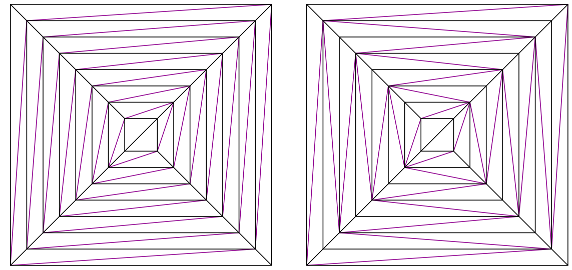
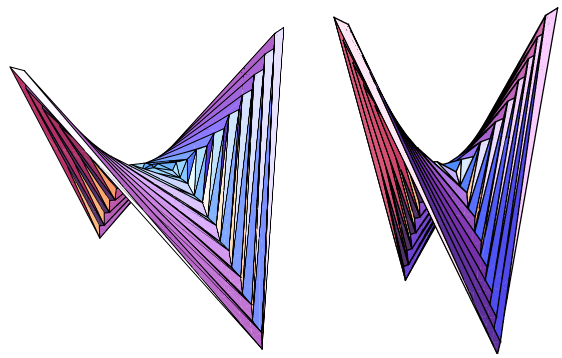


(a) Asymmetric triangulation. (b) Alternating asymmetric triangulation.

Figure 3: Two foldable triangulations of the hyperbolic paraboloid crease pattern (less one diagonal in the center).



(a) Asymmetric triangulation, $\theta = 8°, n = 16$. (b) Alternating asymmetric triangulation, $\theta = 30°, n = 16$.

Figure 4: Proper foldings of triangulated hyperbolic paraboloids.

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