Hinged Dissection of Polyominoes and Polyforms

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Abstract

This paper shows how to hinge together a collection of polygons at vertices in such a way that a single object can be reshaped into any n-omino, for a given value of n. An *n-omino* is defined generally as a connected union of n unit squares on the integer grid. Our best dissection uses $2(n-1)$ polygons. We generalize this result to the connected unions of nonoverlapping equal-size regular k-gons joined edge-to-edge, which includes *n-iamonds* ($k=3$) and *n-hexes* ($k=6$). Our best dissection uses $\lceil k/2 \rceil (n-1)$ polygons. We also explore polyabolos, that is, connected unions of nonoverlapping equal-size isosceles right triangles joined edge-to-edge, and give a hinged dissection using $4n$ polygons. Finally, we generalize our result about regular polygons to connected unions of nonoverlapping copies of any polygon P, all with the same orientation, that join corresponding edges of P. This solution uses kn pieces where k is the number of vertices of P.

1 Introduction

Dissections [5, 10] involve cutting a polygon into several pieces, rearranging them, and gluing them back together to make another polygon. It is well known, for example, that any polygon can be dissected into any other polygon with the same area [2, 5, 11], but the bound on the number of pieces is quite weak. The main problem, then, is to find a dissection with the fewest possible number of pieces.

Frederickson's invited talk at SoCG'98 [6] helped introduce dissections to the computational geometry community. As a result, dissections have begun to be studied more formally than in their recreational past. For example, Kranakis, Krizanc, and Urrutia [9] study the asymptotic number of pieces required to dissect a regular m-gon into a regular n-gon.

Frederickson [6] also mentioned an interesting open problem, *hinged dissections*. Instead of allowing the pieces to be rearranged arbitrarily, suppose that the pieces are hinged together at their vertices. For example, Figure 1 shows the classic hinged dissection of an equilateral triangle into a square. This dissection is described by Dudeney [3], but may have been discovered by C. W. McElroy; see [5, p. 136].

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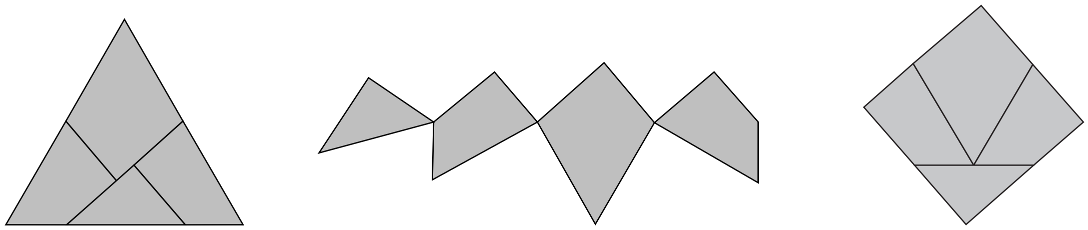

Figure 1: Hinged dissection of an equilateral triangle into a square.

A natural question about hinged dissections is the following: can any polygon be hinge-dissected into any other polygon with the same area? This question is open and seems quite difficult. The main impediment to applying the same techniques as normal dissection is that hinged dissections are not obviously transitive: if A can be hinge-dissected to B, and B can be hinge-dissected to C, then it is not clear how to combine the two dissections into one from A to C. Of course, this transitivity property holds for normal dissections.

As some partial progress towards solving this problem, many examples of hinged dissections have been discovered. Frederickson [4] has developed several techniques for constructing hinged dissections, and has applied them to over 150 examples. Akiyama and Nakamura [1] have demonstrated some hinged dissections under a restrictive model of hinging, designed to match the dissection in Figure 1. For example, they show that it is possible to hinge-dissect any quadrangle into a parallelogram. In general, their work only deals with polygons having a constant number of vertices.

In this paper, we explore hinged dissections of a class of polygons, the connected unions of nonoverlapping regular k-gons for a fixed k, all of the same size, and glued edge-to-edge. We call such a polygon a *poly-k-regular* or *polyregular* for short. The polygon need not be simply connected; we allow it to have holes. An *$n \times k$-regular* is a poly-k-regular made of n regular k-gons.

Poly-k-regulars include the well-studied *polyominoes* ($k = 4$), *polyiamonds* ($k = 3$), and *polyhexes* ($k = 6$) [7, 8, 12]. Polyominoes are of particular interest to computational geometers, because they include orthogonal polygons whose vertices have rational coordinates.

This paper proves that not only can any $n \times k$-regular be hinge-dissected into any other $n \times k$-regular, but furthermore that there is a single hinged dissection that can be folded into any $n \times k$-regular, for fixed n and k. Section 4 describes our first, simpler method, which uses $k(n-1)$ pieces. This method is improved in Section 5 to use $\lceil k/2 \rceil (n-1)$ pieces.

Next, in Section 6 we consider another kind of polyform. A *polyabolo* is a connected union of nonoverlapping isosceles right triangles all of the same size, and joined edge-to-edge. In particular, every n-omino is a $2n$-abolo, as well as a $4n$-abolo. We prove that there is a hinged dissection that can be folded into any n-abolo for fixed n. It uses $4n$ pieces.

Finally, in Section 7, we show an analogous result for a general kind of "polyform," which allows us to take certain unions of copies of a general polygon. This result is a generalization of polyregulars, although it uses more pieces. It does not, however, include the polyabolo result, because of some restrictions placed on how the copies of the polygon can be merged.

2 Basic Structure of Polyforms

We begin with some basic results about the structure of polyforms. As we understand it, the term "polyform" is not normally used in a formal sense, but rather as a figurative term for objects like polyominoes, polyiamonds, polyhexes, and polyabolos. However, in this paper, we find it useful to use a common term to specify all of these objects collectively, and "polyform" seems a natural term for this purpose.

Specifically, we define a (planar) *polyform* to be a finite collection of copies of a common polygon P such that their union is connected, and the intersection of two copies is either empty, a common vertex, or a common edge. An n-form is a polyform made of n copies of P. We call P the *type* of the polyform.

The *dual graph* of a polyform is defined as follows. Create a vertex for each polygon in the collection, and connect two vertices precisely if the corresponding polygons share an edge. Because every graph has a vertex whose removal leaves the graph connected, we have the following immediate consequence.

Lemma 1 *Every n-form has a polygon whose removal results in a (connected) $(n-1)$-form of the same type.*

This simple result is useful for inducting on the number of polygons in a polyform. More precisely, if we view the decomposition in the reverse direction (adding polygons instead of removing them), then this lemma says that any polyform can be built up by a sequence of additions such that any intermediate form is also connected. To construct a hinged dissection, we will repeatedly hinge a new polygon onto the previously constructed polyform. In most cases, this will suffice, but for more complex dissections we will need a more powerful (and indeed more explicit) version of Lemma 1:

Lemma 2 *The polygons in any n-form can be ordered $P_1, \ldots, P_n$ such that for each P_i, we can assign a parent $p(P_i) = P_j$ for some $j < i$. In other words, the n-form can be constructed by starting with P_1, and for $i = 2, \ldots, n$ adding P_i, which is adjacent to (at least) one existing polygon $p(P_i)$. Furthermore, parents can be assigned so that P_2 is the only child of P_1.*

Proof: The lemma is obvious for $n = 1$. Consider an n-form F for $n \geq 2$, and take a spanning tree of its dual graph. Because it has at least two vertices, it must have at least two leaves, call them P_1 and P_n. Root the spanning tree at P_1, thereby defining a parent relation. Repeatedly pick an unnamed vertex whose parent has already been named (which by definition is not P_n), and name it P_i for the next value of i. In the end, every vertex will have a unique name, and the parent of each vertex (except the root P_1) will be named with a smaller index. Furthermore, because P_1 [P_n] is a leaf of the tree, it has a single child [parent]. $\square$

The basic structure of our constructions of hinged dissections will be as follows. First we describe what the pieces are and how they are connected. Second we prove that any polyform can be folded from this dissection using one of the two lemmas. This involves showing how to construct a single polygon (P_1), and then how to add on each polygon P_i to a hinging of an $(i-1)$-form, so that in the end we have a hinged dissection of the entire n-form as originally described.

3 Polyominoes

Let us start with the special case of polyominoes. This serves as a nice introduction to efficient hinged dissection of polyregulars, and is also where our research began.

Constructing a hinged dissection that forms any n-omino is easy for small values of n. There is only one monomino and one domino, so no hinges are necessary. There are two trominoes up to reflection, and a two-piece dissection is easy to find; see Figure 2. Amazingly, a four-piece hinged dissection is possible for tetrominoes, up to reflection; see Figure 3.

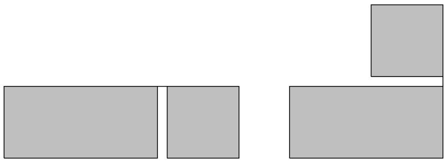

Figure 2: Two-piece hinged dissection of all trominoes, up to reflection.

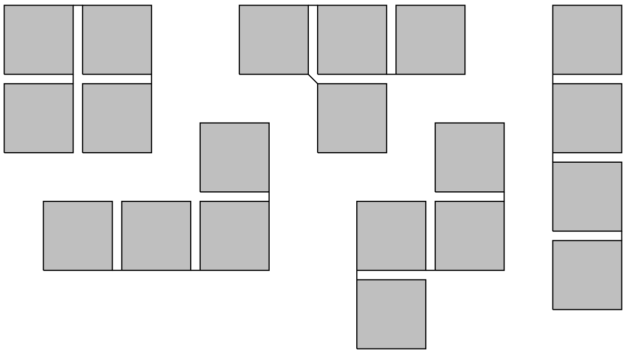

Figure 3: Four-piece hinged dissection of all tetrominoes, up to reflection.

Unfortunately, it seems that for $n > 4$, dividing an n-omino into its constituent squares is insufficient for it to hinge into all other n-ominoes. (This is in contrast to normal dissections, where it is clearly sufficient.)

All of our future hinged dissections will not rely on the "up to reflection" proviso, and only require folding, translation, and rotation to form the desired polyform. Our first, simplest hinged dissection of n-ominoes uses $2n$ isosceles right triangles; see Figure 4. Note that the cycle dissection is a stronger result: simply breaking one of the hinges results in the path dissection.

Theorem 1 *A cycle of $2n$ isosceles right triangles, joined at their base vertices, can be folded into any n-omino.*

Proof: The proof is by induction on n. The case $n = 1$ is trivial; see Figure 5.

Now suppose that the theorem holds for all $n < k$, where $k > 1$. Let P be any k-omino. We will show that the hinging of $2k$ triangles can be folded into P. By Lemma 1, there is a square S of P such that $P - S$ is a polyomino. By induction, $2k - 2$ hinged triangles can be folded into $P - S$. Now we extend this to $2k$ hinged triangles folded into P; refer to Figure 6.

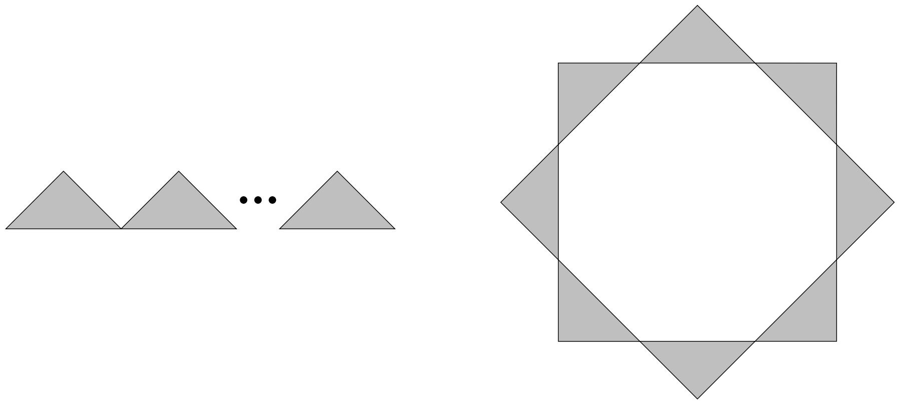

Figure 4: $2n$-piece hinged dissection of all n-ominoes. (Left) Connected in a path. (Right) Connected in a cycle, $n = 4$.

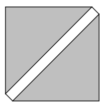

Figure 5: The case $n = 1$: a 2-piece hinged dissection of a monomino.

Let T be a triangle in this hinging that shares an edge with S. One of its base vertices, say v, is also incident to S, and it must be hinged to some other triangle T'. We split S into two right triangles S_1 and S_2 so that both have a base vertex at v. Now we replace T's hinge at v with a hinge to S_1, and add a hinge from S_2 to T' at v. Finally, S_1 and S_2 are hinged together at their other base vertex. The result is a hinging of $2k$ triangles folded into P. We can optionally swap S_1 and S_2 in order to avoid crossings between the hinges. $\square$

Now we explain how to modify this dissection to use two fewer pieces:

Corollary 1 *For $n \geq 2$, the $(2n-2)$-piece hinged dissection in Figure 7 can be folded into any n-omino.*

Proof: Consider the last square $S = (S_1, S_2)$ added in the construction of an n-omino by Theorem 1. Disconnect the cycle of hinges into a path by breaking the hinge between S_2 and T'. Now T, S_1, and S_2 are hinged together to form the rightmost shape in Figure 7 (a union of three isosceles right triangles). Because this holds for any n-omino, we can join these three triangles in the hinged dissection. $\square$

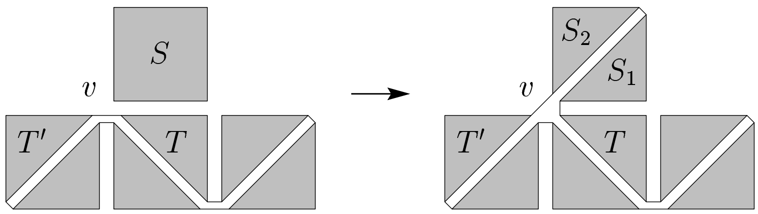

Figure 6: Adding a square S to the $2n$-piece hinged dissection of n-ominoes.

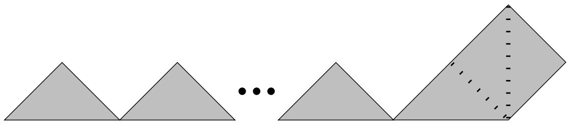

Figure 7: $(2n-2)$-piece hinged dissection of all n-ominoes.

This method will be generalized in Section 5 to support all polyregulars, with a matching bound on the number of pieces for $k = 4$.

4 Polyregulars

This section and the next give methods for constructing, given any $n \geq 1$ and $k \geq 3$, a hinged dissection that can be folded into all $n \times k$-regulars. In particular, for $k = 4$, this solves the polyomino case. However, the solution in this section will not be as efficient as that in the previous section. Later we will see how to combine the two methods to result in a single method that is more efficient for all k. We present two methods for polyregulars because the one in this section, while using more pieces, is simpler and eases the understanding of the method in the next section.

Our first hinged dissection splits each regular k-gon into k isosceles triangles, by adding an edge from each vertex to the center of the polygon. See Figure 8. Because it is rather difficult to draw a generic regular k-gon, our figures will concentrate on the case of $k = 3$, i.e., n-iamonds. We define the *base* of each isosceles triangle to be the edge whose length differs from all the others, the *base angles* to be the interior angles of the incident vertices, and the *opposite angle* to be the interior angle of the remaining vertex.

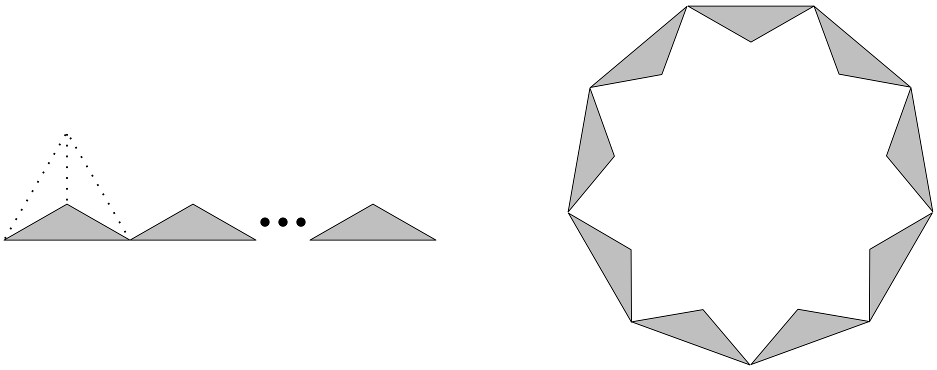

Figure 8: $3n$-piece hinged dissection of all n-iamonds. (Left) Connected in a path. (Right) Connected in a cycle, $n = 3$.

Theorem 2 *A cycle of kn isosceles triangles with opposite angle $2\pi/k$, joined at their base angles, can be folded into any $n \times k$-regular.*

Proof: The proof is by induction on n. The case $n = 1$ is trivial; see Figure 9.

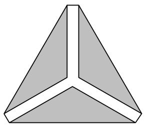

Figure 9: The case $n = 1$: a 3-piece hinged dissection of a moniamond.

Now suppose that the theorem holds for all smaller values of n, where $n > 1$. Let P be any $n \times k$-regular. We will show that the hinging of kn triangles can be folded into P. By Lemma 1, there is a regular k-gon R of P such that $P - R$ is a polyregular. By induction, $k(n-1)$ hinged triangles can be folded into $P - R$. Now we extend this to kn hinged triangles folded into P; refer to Figure 10.

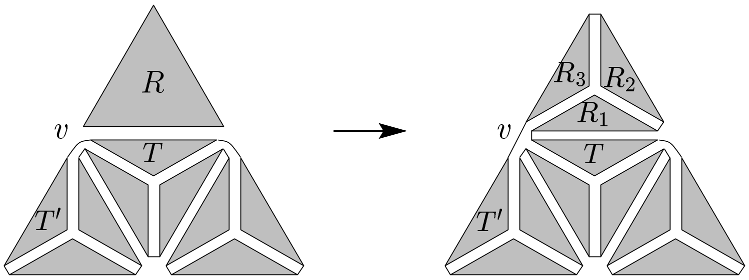

Figure 10: Adding an equilateral triangle R to the $3n$-piece hinged dissection of n-iamonds.

Let T be a triangle in this hinging that shares an edge with R. Both of its base vertices are also incident to R. Let v be one of the base vertices, and suppose that it is hinged to triangle T'. We split R into k triangles $R_1, \ldots, R_k$ so that both R_1 and R_k have a base vertex at v. Now we replace T's hinge at v with a hinge to R_1, and add a hinge from R_k to T' at v. Finally, R_i and R_{i+1} are hinged together at their common base vertex, for all $1 \leq i < k$. The result is a hinging of kn triangles folded into P. We can optionally renumber $R_1, \ldots, R_k$ as $R_k, \ldots, R_1$ in order to avoid crossings between the hinges. $\square$

While the number of pieces will be improved dramatically in the next section, we show that the trick of merging the last few pieces also applies to this dissection, reducing the number of pieces by k:

Corollary 2 *For $n \geq 2$, the $k(n-1)$-piece hinged dissection in Figure 11 can be folded into any $n \times k$-regular.*

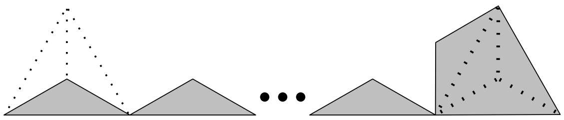

Figure 11: $(3n-3)$-piece hinged dissection of all n-iamonds.

Proof: Consider the last regular k-gon $R = (R_1, \ldots, R_k)$ added in the construction of an $n \times k$-regular by Theorem 2. Disconnect the cycle of hinges into a path by breaking the hinge between R_k and T'. Now T, R_1, ..., R_k are hinged together to form the rightmost shape in Figure 11 (a union of $k+1$ isosceles triangles). Because this holds for any $n \times k$-regular, we can join these $k+1$ triangles in the hinged dissection. $\square$

5 Improved Polyregulars

The goal of this section is to improve the hinged dissection for polyregulars in Section 4 so that, for $k = 4$, the number of pieces matches the method in Section 3 for polyominoes. To see how to do this, let us compare the two methods when restricted to $k = 4$. The method in Section 4 splits each square into four right isosceles triangles; i.e., it makes four cuts to the center of the square. In contrast, the method in Section 3 makes only two of these cuts. In other words, the method in Section 3 can be thought of as merging adjacent pairs of right isosceles triangles from the method in Section 4.

This suggests the following generalized improvement to the method in Section 4, for arbitrary k: join adjacent pairs of right isosceles triangles, until zero or one triangles are left. For even k (like $k = 4$), this will halve the number of pieces; and for odd k, it will almost halve the number of pieces. The intuition behind why this method will work is that when we added a regular k-gon to an existing polyregular in the proof of Theorem 2, we had two existing hinges at which we could connect the new k-gon; at most halving the number of hinges will still leave at least one hinge to connect the new k-gon.

In general, our hinged dissection will consist of $\lceil k/2 \rceil n$ pieces. If k is even, every piece will be the union of two isosceles triangles, each with opposite angle $2\pi/k$, joined along an edge other than the base. If k is odd, every group of $\lfloor k/2 \rfloor$ of these pieces is followed by a single isosceles triangle with opposite angle $2\pi/k$. For example, for polyiamonds ($k = 3$), the pieces alternate between single triangles and "double" triangles (see Figure 12). Independent of the parity of n, the pieces are joined at base vertices of one of the constituent triangles that is not merged to another base vertex.

Again, our figures will focus on the case $k = 3$, as in Figure 12.

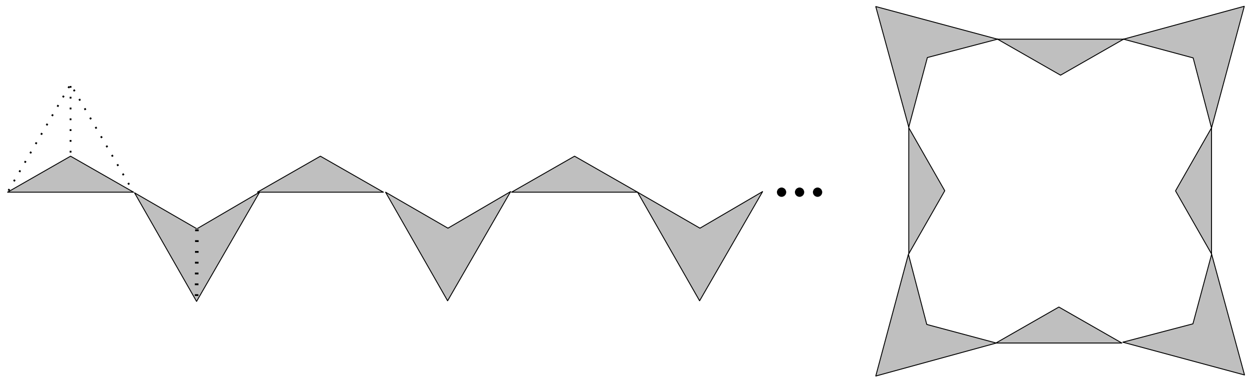

Figure 12: $2n$-piece hinged dissection of all n-iamonds. (Left) Connected in a path. (Right) Connected in a cycle, $n = 4$.

Theorem 3 *The described hinged dissection of $\lceil k/2 \rceil n$ pieces connected in a cycle can be folded into any $n \times k$-regular.*

Proof: The proof is by induction on n. The case $n = 1$ is trivial; see Figure 13.

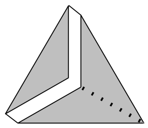

Figure 13: The case $n = 1$: a 2-piece hinged dissection of a moniamond.

Now suppose that the theorem holds for all smaller values of n, where $n > 1$. Let P be any $n \times k$-regular. We will show that the hinging of $\lceil k/2 \rceil n$ pieces can be folded into P. By Lemma 1, there is a regular k-gon R of P such that $P - R$ is a polyregular. By induction, $\lceil k/2 \rceil (n-1)$ hinged pieces can be folded into $P - R$. Now we extend this to $\lceil k/2 \rceil n$ hinged pieces folded into P; refer to Figure 14.

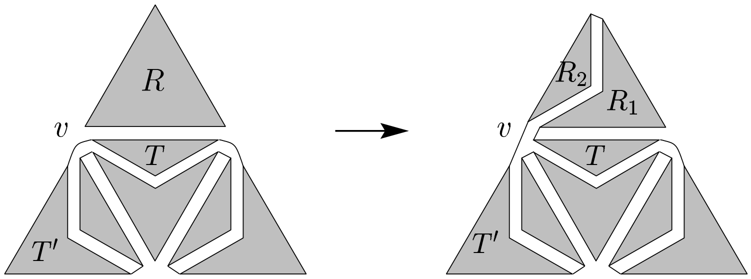

Figure 14: Adding an equilateral triangle R to the $2n$-piece hinged dissection of n-iamonds.

Let T be a piece in this hinging that shares an edge with R. Both of the base vertices of one of its constituent triangles are incident to R. Let v be such a base vertex of T that is not joined to a base vertex of another constituent triangle of T (which we call *lone* base vertices), and suppose that it is hinged to piece T'. We split R into $\lceil k/2 \rceil$ pieces $R_1, \ldots, R_{\lceil k/2 \rceil}$ so that both R_1 and $R_{\lceil k/2 \rceil}$ have a lone base vertex at v. Now we replace T's hinge at v with a hinge to R_1, and add a hinge from $R_{\lceil k/2 \rceil}$ to T' at v. We hinge R_i and R_{i+1} together at their common lone base vertex, for all $1 \le i < \lceil k/2 \lceil$. Finally, if k is odd, we choose one of the pieces $R_1, \ldots, R_{\lceil k/2 \rceil}$ to be a single triangle instead of a double triangle, appropriately so that the resulting cycle of pieces has the desired uniform distribution of single triangles. The result is a hinging of $\lceil k/2 \rceil n$ pieces folded into P. We can optionally renumber $R_1, \ldots, R_{\lceil k/2 \rceil}$ as $R_{\lceil k/2 \rceil}, \ldots, R_1$ in order to avoid crossings between the hinges. $\square$

Our final hinged dissection of polyregulars improves the previous one by $\lceil k/2 \rceil$ pieces.

Corollary 3 *If we break the cycle of the previous dissection and merge the first $\lceil k/2 \rceil + 1$ pieces (see Figure 15), then we have a hinged dissection of all $n \times k$-regulars using $\lceil k/2 \rceil (n-1)$ pieces for $n \ge 2$.*

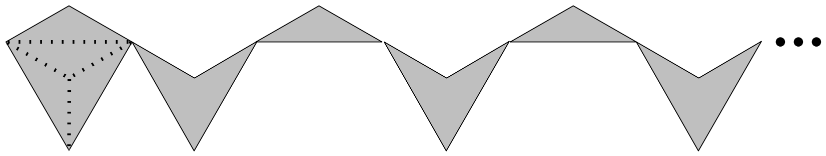

Figure 15: $(2n-2)$-piece hinged dissection of all n-iamonds.

Proof: Note that we cannot simply merge the last few pieces as we have in all previous corollaries, because then the placement of the regular k-gon may vary with respect to the other pieces in the dissection. That is, the distance between the k-gon and the lone triangle (if k is odd) might differ depending on the polyregular and the building sequence.

To solve this problem, we choose a building sequence according to Lemma 2. This means that the *first* regular k-gon also only connects to one other k-gon. Thus, we can merge the pieces of the first regular k-gon together with one piece of the second one, and then build on from there, choosing the positioning of the first single triangle in a consistent way relative to the big merged piece. More precisely, our base case of $n = 2$ is modified to make one big piece for the first k-gon combined with several pieces for the second k-gon, where the latter several pieces are chosen so that the next single triangle (for k odd) is the $\lfloor k/2 \rfloor$nd piece after the big piece. The remaining steps in the induction proceed as before. Because the order of insertion of k-gons is chosen so that no future k-gon will attach to the first one, this hinged dissection is valid. $\square$

6 Polyabolos

Another well-studied class of polyforms that does not fall under the class of polyregulars is *polyabolos*, the union of equal-size half-squares (right isosceles triangles) joined at edges. In this section, we present hinged dissections of polyabolos.

Our first dissection is a cycle of $4n$ right isosceles triangles, as shown in Figure 16. Like Figure 4, the triangles point outward, but unlike Figure 4, they are joined at a short edge instead of the long edge. The orientations of the triangles (or equivalently, which of the two short edges we connect the triangle to the others) alternate along the cycle.

Theorem 4 *The $4n$-piece hinged dissection in Figure 16 can be folded into any n-abolo.*

Proof: The proof is by induction on n. The case $n = 1$ is trivial; see Figure 17.

Now suppose that the theorem holds for all $n < k$, where $k > 1$. Let P be any k-abolo. We will show that the hinging of $4k$ triangles can be folded into P. By Lemma 1, there is a half-square H of P such that $P - H$ is a polyabolo. By induction, $4k - 4$ hinged triangles can be folded into $P - H$. Now we extend this to $4k$ hinged triangles folded into P, using three different cases depending on the position of H relative to an incident half-square; see Figure 18. In general, we just take a hinging of the single half-square H to be added, as in Figure 17, and merge it with the unique incident hinge. It is easy to verify from Figure 18 that this always keeps the triangles pointing outward from the cycle, and alternating in orientation along the cycle. $\square$

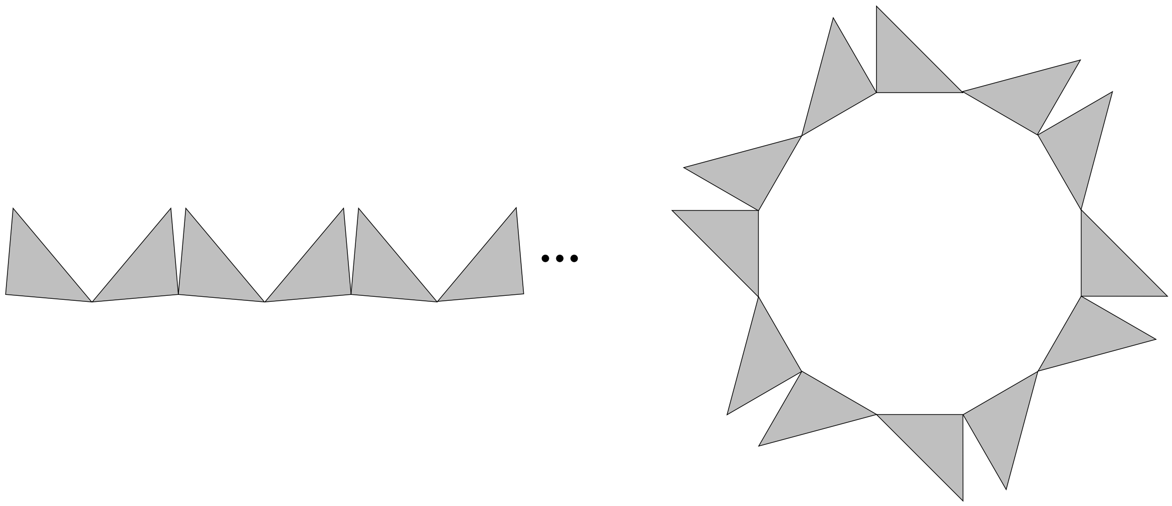

Figure 16: $4n$-piece hinged dissection of all n-abolos. (Left) Connected in a path. (Right) Connected in a cycle, $n = 3$.

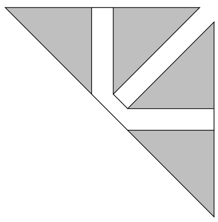

Figure 17: The case $n = 1$: a 4-piece hinged dissection of a monabolo.

An interesting consequence of the previous theorem is a hinged dissection that can be folded into polyominoes of different sizes:

Corollary 4 *A common hinged dissection can be folded into any n-omino and any $2n$-omino.*

Proof: Either can be viewed as a $4n$-abolo, by splitting a square in the n-omino into four pieces (Figure 19, left) and splitting a square in the $2n$-omino into two pieces (Figure 19, right). □

This dissection uses a large number of pieces, namely $16n$. In fact, we can do much better by simply taking the $4n$-piece path dissection of the $2n$-omino from Figure 4, left. The full cyclic hinging does not work, because the $2n$-omino wants the long sides inside while the n-omino wants them outside, so folding from one to the other would require "twisting" the hinges.

7 Other Polyforms

An interesting open problem is whether there is a hinged dissection that can be folded into any polyform. In other words, for a fixed n and polygon P, is there a hinged dissection

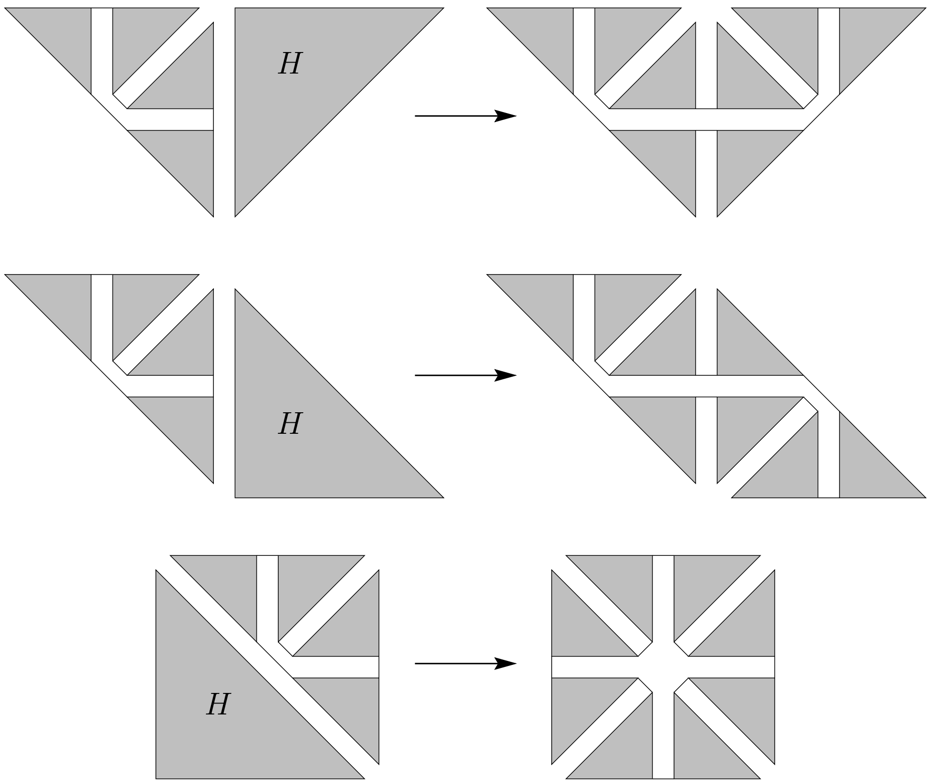

Figure 18: Adding a half-square H to the $4n$-piece hinged dissection of n-abolos.

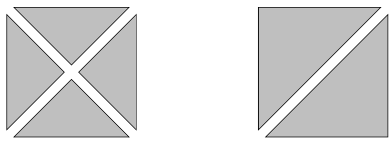

Figure 19: Two ways to split a polyomino into a polyabolo.

that folds into any connected union of n copies of P joined edge-to-edge? As a step towards solving this problem, we present a hinged dissection for restricted polyforms of type P. Specifically, we impose the restriction that for any two copies of P sharing an edge, there must be a rigid motion that

1. takes one copy of P to the other copy, and

2. takes the shared edge in one copy to the shared edge in the other copy.

The first constraint says that all copies of P have the same clockwise/counterclockwise orientation. This may be included in the notion of "polyform," depending on your definition. The second constraint says that only "corresponding edges" of copies of P are joined. This is actually not that uncommon: if P is generic in the sense that no two edges have the same length, then the second constraint is implied by the edge-to-edge condition.

Comparing to our previous results, every polyregular is a polyform satisfying the described restriction. However, polyabolos do not satisfy the second component; for example, if we join two right isosceles triangles so that their union forms a larger right isosceles triangle, then noncorresponding edges have been joined.

The method for dissecting these "restricted polyforms" works as follows. We subdivide P by making cuts incident to the midpoint of every boundary edge, so that there is one piece surrounding each vertex of P. This can be done as follows; refer to Figure 20. Take a triangulation T of P. First, cut along the dual tree D of T, where the vertices of D are positioned somewhere interior to the triangles in T. Second, cut from each vertex of D to the midpoint of every incident edge of P.

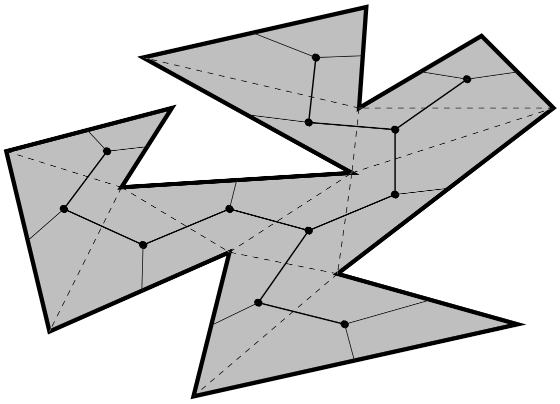

Figure 20: Cutting up a polygon P with cuts through the midpoint of every edge. Only the dashed lines are not cuts; they show the underlying triangulation T. The thick solid lines and dots form the dual tree D, and the thin solid lines are the cuts from dual vertices to edge midpoints.

Now the actual hinged dissection is simple: repeat the cyclic decomposition of P, n times, and hinge the pieces at the midpoints of the edges of P. Now at any edge of P we can decide to visit an incident copy of P before completing the traversal of P, and we visit the same sequence of pieces. See Figure 21 for a simple example. Thus, we can construct any restricted polyform of type P, proving the following theorem:

Theorem 5 *There is a kn-piece hinged dissection that can be folded into any restricted polyform of type P, where P is a polygon with k vertices.*

8 Conclusion

Our most general result is that, for any polygon P and positive integer n, there is a hinged dissection that can be folded into any arrangement of n copies of P with fixed orientation joining at corresponding edges. In particular, this included polyregulars (equal-size regular polygons joined edge-to-edge) as a subclass, for which we showed how to improve the number of pieces. This class contained as subclasses several well studied objects: polyominoes (equal-size squares joined edge-to-edge), polyiamonds (equal-size equilateral triangles joined edge-to-edge), and polyhexes (equal-size regular hexagons joined edge-to-edge). Finally, we proved

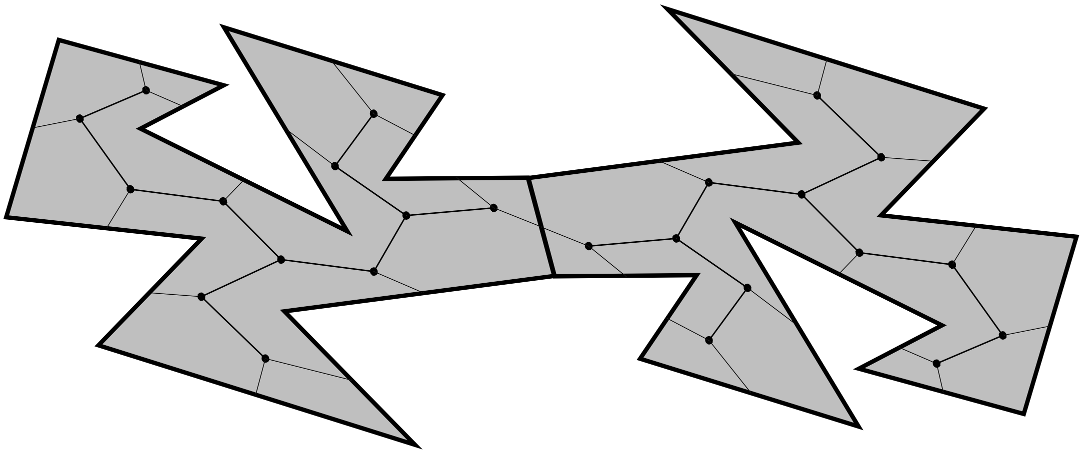

Figure 21: Joining two copies of P, once each is cut up: switching over from one copy of P to the other does not affect the order of shapes of pieces we visit.

the analogous result for polyabolos (equal-size right isosceles triangles joined edge-to-edge), which do not fall under any of the above classes, but are still considered "polyforms."

Let us conclude with a list of interesting open problems about hinged dissections, focusing on polyforms:

1. Can our results be generalized to arbitrary polyforms, that is, connected unions of n nonoverlapping copies of a common polygon joined edge-to-edge?

2. How many pieces are needed for a hinged dissection of all pentominoes? What about general n-ominoes as a function of n? We know of no nontrivial lower bounds.

3. Can different regular polygons be mixed? For example, can any regular k-gon be hinge-dissected into any regular k'-gon with the same area? Figure 1 shows this is true for $\{k, k'\} = \{3, 4\}$.

4. Can any n-omino be hinge-dissected into any m-omino (of an appropriate scale), for all n, m? In Section 6, we proved this is true for $m = 2n$.

5. We have shown that there exist configurations of a common hinged dissection in the form of any $n \times k$-regular. Is it possible to continuously move the dissection from one configuration to another, while keeping the pieces nonoverlapping?

6. It would be interesting to generalize to higher dimensions. For example, *polycubes* are connected unions of nonoverlapping unit (solid) cubes joined face-to-face. Can a collection of solids be hinged together at edges so that the dissection can be folded into any n-cube (for fixed n)?

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