On Reconfiguring Tree Linkages: Trees can Lock

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Abstract

It is an open problem to determine whether a polygonal chain can be "straightened" in the plane if its links are not allowed to cross. This problem been raised independently by several researchers, including J. Mitchell, and W. Lenhart and S. Whitesides [LW95]. In this paper we propose a related question: whether a *tree linkage* can always be "straightened" in the plane, without allowing its links to cross. We prove that this is not always possible.

1 Introduction

Consider a graph, each edge labelled with a positive number. Such a graph may be thought of as a collection of distance constraints between pairs of points in a Euclidean space. A *realization* of such a graph maps each vertex to a point, also called a *joint*, and maps each edge to the closed line segment, called a *link*, connecting its incident joints. The link length must equal the label of the underlying graph edge. If a graph has one or more such realizations, we call it a *linkage*. An embedding of a linkage in space is called a *configuration* of the linkage. The linkage can move from one configuration to another, as long as the link lengths are respected throughout the motion. In a *simple configuration*, a pair of links whose underlying edges are incident on a common vertex intersect only at the common joint, and all other pairs of links are disjoint.

The *linkage reconfiguration* problem is as follows: given two configurations, can the linkage move from one to the other? Previous work (e.g. [LW95]) has considered linkages whose underlying graph is a path or a simple cycle; we call such linkages *chain* and *cycle* linkages, respectively. Most previous reconfiguration results for chain and cycle linkages allow the links to cross or to pass through or over one another; in other words, in previous results, the links represent distance constraints between joints, not physical obstacles that must avoid each other.

When links are not allowed to cross, as is the case for this paper, there are two natural definitions for the kind of movements allowed. The first definition, which we call a *free* motion, forbids any contact between two links except at the common joint of two links whose corresponding graph edges are adjacent. The second definition, which we call *semifree* motion, is less restrictive than the first:

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a semifree motion is a motion that allows a joint of one link to touch another link, provided it does not cross over it. We give a more technical definition below.

Even the most basic questions concerning the reconfiguration of linkages, whether moving under free or semifree motion, have proved difficult to solve. For example, the question of whether a chain linkage in the plane, moving under either free or semifree motion, can always be reconfigured to a stretched out configuration in which adjacent links form angles of measure π, has been open for several years [LW95]. Similarly, it is unknown whether a cycle linkage can always be reconfigured to form a convex polygon. In the case that the initial embedding of the linkage forms a star-shaped polygon, i.e., some point inside the polygon sees the entire boundary of the polygon, a free motion that convexifies the linkage always exists [ELR$^+$98].

This paper defines and explores a reconfiguration problem for linkages in the plane, where the underlying graph of the linkage is a tree. We call such a linkage a *tree linkage*. This problem was originally proposed in [DD97], where it is motivated by a (two-dimensional) paper-folding problem of lining up the boundary of a polygon. The initial configuration for the linkage is simple, and the linkage moves under semifree motion. Thus, links are not allowed to pass over one another, but joints may touch and slide along links to which they are not joined.

The basic problem we consider is whether all configurations of a tree linkage are reachable, one from another, under semifree motion. Our main result is that this is not possible in the general case. We obtain this result by proving that tree linkages starting in a simple configuration (sketched in Figure 1) cannot necessarily be *straightened*, that is, moved to a *straightened configuration*. A straightened configuration is one in which the links all lie on a ray emanating from one of the joints, with angles formed between "parent" and "child" links measuring π. We define this term more precisely in the next section.

2 Preliminaries

In this section, we introduce the technical definitions we use. From now on, we often do not distinguish between vertices and edges in the underlying tree, the corresponding joints and links in the tree linkage, and the points and line segments occupied by the joints in links in a particular configuration. The context should make the meaning clear.

Definition: Choose a vertex r to be the root of a tree. For any vertex v, an incident link that joins v with its parent is called a *parent link*; otherwise it is called a *child link*. A *straightened configuration* of a tree linkage is one in which the parent link at each non-leaf joint makes angle π with each child link at that joint, and the angle between each pair of child links is zero. To *straighten* a linkage configuration means to move it to a straightened configuration. We say a linkage configuration is *locked* if it cannot be straightened.

Thus, a straightened linkage lies on a ray emanating from some joint that can be regarded as a root for the underlying tree. One can show that if a tree linkage configuration can be straightened, then it can be straightened using any joint as the root.

Next, we define more precisely what motions we allow.

Definition: A *motion* M is a map taking the set of vertices, V, into the plane, as a continuous function of time. We require the motion to begin at time 0 and end at time 1; thus $M : V \times [0,1] \to E^2$. Furthermore, the distance constraint between adjacent joints given by the edge label must be respected over the entire time interval.

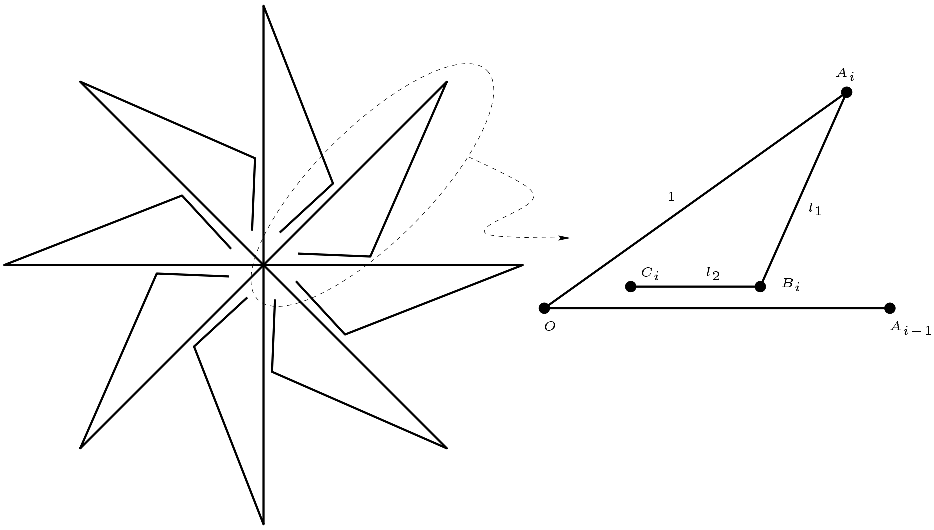

Figure 1: An example linkage with enlarged view of petal i, showing the joint labels and the link lengths.

Definition: A motion M is *free* provided all configurations of the linkage are simple throughout the time interval of the motion.

Given two points in the plane, X and Y, we use XY to designate the closed line segment between X and Y, and $\|XY\|$ to denote the length of the segment XY.

Definition: A motion M is *semifree* if, for any $\epsilon > 0$, there exists a free motion M' within distance ϵ of M, where the distance between motions M and M' is $\max_{v \in V,\, 0 \le t \le 1} \|M(v,t)M'(v,t)\|$.

The trees considered in this paper consist of n *petals*, each comprised of three links. The initial configuration is sketched in Figure 1 and detailed as the paper unfolds.

The petals all meet at joint O; the other joints of petal i are labelled A_i, B_i, and C_i. Designate the *petal angle* as $\theta_i = \angle A_{i-1}OA_i$; let $\bar{\theta} = 2\pi/n$. All index arithmetic is taken mod n; all angles are measured in the interval $[0, 2\pi)$. The link lengths are chosen as follows: $\|OA_i\| = 1$, $\|A_iB_i\| = l_1$, and $\|B_iC_i\| = l_2$.

We often focus on a single petal at a time, so the notation is simplified by suppressing the petal index. Joints of petal i are referred to as A, B, and C, joint A_{i-1} is denoted A', and the petal angle, i.e., $\angle A'OA$, as just θ. Let L be the line through OA, and let L' be the line through OA'. Choose a reference frame with L' oriented horizontally, O to the left of A'.

3 The Intuition

We pause to give a brief outline of the main argument, before delving into the details.

In the initial configuration, all petals are in a congruent configuration, pictured in Figure 2, with petal angle $\bar{\theta}$. By choosing l_1 long enough, we ensure that link AB can not swing out so as to straighten joint A. Thus, the petal angle must be increased before joint A can straighten. Because the petals all join at O, opening up one petal necessitates squeezing the other petals.

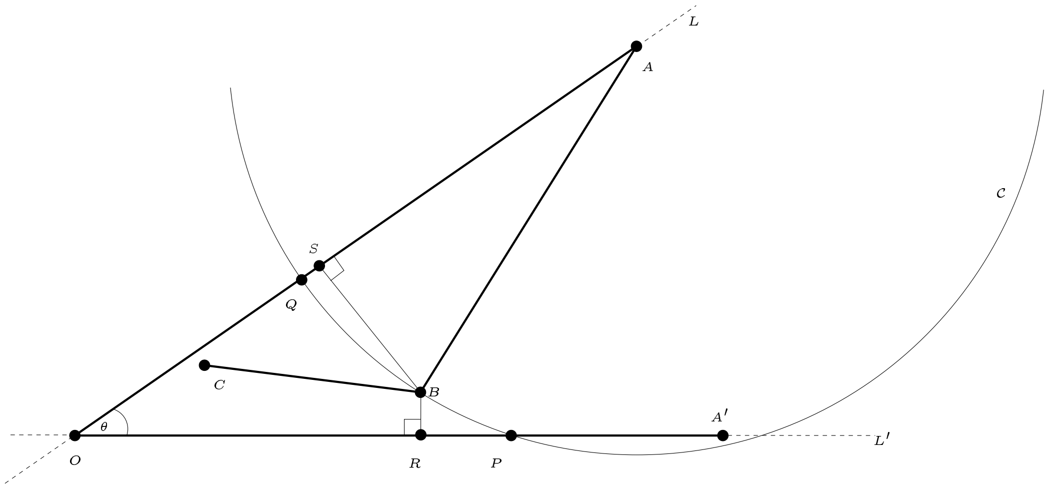

Figure 2: Illustrating the definition of *restricted configuration*.

Consider now how small we can squeeze a petal angle. By choosing l_2 long enough, we ensure that link BC can not swing over to fold up against link AB. In fact, joint C is trapped inside quadrilateral $\square ORBS$. We show later that the smallest petal angle is obtained by moving joint C to O, pictured in Figure 3. Lengthening l_2 increases this minimum petal angle, so by choosing l_2 long enough, we can ensure that squeezing $n-1$ petals to their minimum angle is still not enough to let the last petal open. This, in essence, is why the tree is locked.

From the above discussion, we note that l_1 and l_2 must be chosen "long enough". On the other hand, they must also be "short enough" that the simple configuration show in Figure 2 is achievable. Section 4 details all the constraints needed to satisfy these two requirements. Section 5 defines a "restricted" class of petal configurations (to which the configuration in Figure 2 belongs) and proves that there is a non-zero minimum petal angle for this class of configurations. Finally, Section 6 proves that a linkage with parameters n, l_1, and l_2, satisfying all the constraints of Section 4, can be put into a simple initial configuration that is locked.

4 Constructing a Locked Tree

This section details the constraints on the parameters n, l_1, and l_2 that are necessary to construct a locked linkage, and demonstrates that the constraints are simultaneously satisfiable. Refer to Figure 2 throughout this section.

For $\theta \geq \pi/2$, there is no hope of locking a three-link petal. Thus we must have $\bar{\theta}$, the initial petal angle, strictly less than $\pi/2$, or

$$n > 4. \tag{1}$$

Henceforth, assume $\theta < \pi/2$. We want to have joint B to the left of the vertical line through A. In order that link AB fits, we must have

$$l_1 < 1. \tag{2}$$

For any configuration of a petal, let $\mathcal{C}$ be the circle centred at A with radius l_1. Let β be the petal angle at which $\mathcal{C}$ is tangent to L':

$$\beta = \arcsin l_1. \tag{3}$$

Circle $\mathcal{C}$ will properly intersect L' as long as $\theta < \beta$. Given Inequality 2, we note that $\beta < \pi/2$.

Suppose $\mathcal{C}$ properly intersects L' and let P be the leftmost intersection point. Applying the cosine rule to $\triangle OAP$ yields

$$\|AP\|^2 = \|OP\|^2 + \|OA\|^2 - 2 \cdot \|OP\| \cdot \|OA\| \cos\theta. \tag{4}$$

Since l_1, the radius of $\mathcal{C}$, is strictly less than $\|OA\| = 1$ by Inequality 2, joint O is outside the circle $\mathcal{C}$. From this, and $\theta < \pi/2$, it follows that P is to the right of O, and hence $\|OP\|$ is the smaller root of Equation 4. Noting that $\|AP\| = l_1$, we define

$$l_P(\theta) = \|OP\| = \cos\theta - \sqrt{\cos^2\theta - (1 - l_1^2)}. \tag{5}$$

Note, for future reference, that one can show that l_P is an increasing function of θ in $[0, \beta]$.

Furthermore, P is to the left of the vertical line through A, and hence $P \in OA'$, but P is not at A'. We know that A is inside the circle $\mathcal{C}$, while O is outside, so $\mathcal{C}$ intersects OA; let Q be this intersection point. Note that the small arc PQ of $\mathcal{C}$ lies inside $\triangle A'OA$.

Suppose $\theta < \beta$, and B is on the small arc PQ of $\mathcal{C}$. Then B is inside $\triangle A'OA$. Since the triangle is isosceles, and the angle θ is $< \pi/2$, $\triangle A'OA$ is acute. This ensures that there is a line passing through B perpendicular to each edge of the triangle. Let $R \in OA'$ be such that $BR \perp OA'$ and $S \in OA$ be such that $BS \perp OA$. We want to have joint C inside quadrilateral $\square ORBS$. This is feasible for the initial configuration if we choose

$$l_2 < l_P(\bar{\theta}), \tag{6}$$

for then we may obtain a simple configuration by placing B near P, and C along OB. At the same time, we wish to have l_2 long enough that C remains trapped in $\square ORBS$. This is ensured by choosing

$$l_2 > \sin\beta\cos\beta, \tag{7}$$

as we show later in Lemma 3.

Define

$$\alpha = (2\pi - \beta)/(n-1). \tag{8}$$

Later, we will see that if a petal angle is opened past β, then some other petal angle is smaller than α. The proof that the tree is locked then hinges on showing that there is a minimal petal angle, which is greater than α.

To obtain a minimal petal angle, we require

$$l_1 + l_2 > 1, \tag{9}$$

for otherwise, links AB and BC could fold flat against link OA, and the petal angle could squeeze to zero. We will show, in fact, that the minimum possible petal angle is obtained in the configuration with C at O, and $B \in OA'$, pictured in Figure 3. Define θ_m to be the resulting petal angle. With $\triangle OAB$, the cosine rule yields,

$$l_1^2 = l_2^2 + 1 - 2l_2\cos\theta_m,$$

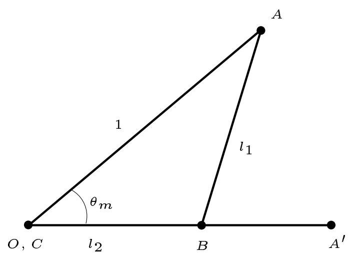

Figure 3: For calculating θ_m.

or

$$\theta_m = \arccos\left(\frac{1 - l_1^2 + l_2^2}{2l_2}\right). \tag{10}$$

We also require l_1 and l_2 to be such that $n\theta_m < 2\pi$, or in other words

$$\theta_m < \bar{\theta}, \tag{11}$$

in order that n petals with $\theta_i \geq \theta_m$ fit around the circle.

Finally, we assume in order to get a contradiction, that l_1 and l_2 are such that

$$\alpha < \theta_m. \tag{12}$$

We now demonstrate that all the constraints of this section may be simultaneously satisfied, by an appropriate choice of n, l_1, and l_2.

Lemma 1 *There exists a positive integer n, and two positive real numbers l_1, and l_2 satisfying simultaneously all the constraints of Section 4 (i.e., Inequalities 1, 2, 6, 7, 9, 11, and 12).*

Proof: We exhibit a choice of parameters satisfying all the constraints. Let $n = 12$, $(\bar{\theta} = 30°)$ $l_1 = 0.61$, and $l_2 = 0.51$.

Inequalities 1 and 2 are immediate. Equation 3 gives $\beta \approx 37.6°$. Equation 5 gives $l_P(\bar{\theta}) \approx 0.52$, satisfying Inequality 6. We have $\sin\beta\cos\beta \approx 0.48$, so Inequality 7 is satisfied. Equation 8 gives $\alpha \approx 29.3°$. Inequality 9 is true by inspection. By Equation 10, we have $\theta_m \approx 29.5°$, satisfying Inequalities 12 and 11. □

Solutions for other choices of n have been found. For example, $n = 5$, $l_1 = 0.9511$, and $l_2 = 0.299$ works. We conjecture that a solution exists for all $n > 4$. In what follows, we assume some choice of parameters such that all the constraints of this section hold.

5 Restricted Configurations

This section is devoted to showing that θ_m is the lower limit for the petal angle.

Referring to Figure 2, recall from the definition of β (Equation 3) that $\theta < \beta$ implies that circle $\mathcal{C}$ properly intersects link OA'. Thus, points P and Q are well-defined. When joint B is on the small arc PQ, recall that points R and S are also well-defined. And finally, joint C also fits inside $\Box ORBS$, due to Inequality 6.

Definition 2 (Restricted Configuration) *A petal configuration is said to be restricted if the following conditions hold*

(i) $\theta < \beta$

(ii) *B is on the closed small arc PQ of $\mathcal{C}$*

(iii) *C is in the closed region bounded by the quadrilateral $\square ORBS$*

Note in Figure 2 that $\|BR\|$ and $\|BS\|$ are both smaller than $\|BC\|$. We show now that this is always the case in a restricted configuration.

Lemma 3 *In a restricted configuration, both $\|BR\|$ and $\|BS\|$ are strictly less than l_2.*

Proof: For a point X and a line L, let $d(X, L)$ denote the distance from X to the line L. We have that $\|BR\| = d(B, L')$. Because B is on the small arc PQ, $d(B, L') \leq d(Q, L') = \|OQ\| \sin\theta$. Hence,

$$\|BR\| \leq \|OQ\| \sin\theta. \tag{13}$$

By similar reasoning,

$$\|BS\| \leq \|OP\| \sin\theta. \tag{14}$$

Note that $\|OQ\| = 1 - l_1 = 1 - \sin\beta$, by Equation 3. For $x \in [0, \pi/2]$, $\sin x + \cos x \geq 1$, so $\|OQ\| \leq \cos\beta$. Because l_P is an increasing function, $\|OP\| = l_P(\theta) \leq l_P(\beta)$, and $l_P(\beta) \leq \cos\beta$, by Equation 5.

Both $\|OP\|$ and $\|OQ\|$ are $\leq \cos\beta$. Hence, by Inequalities 13 and 14, both $\|BR\|$ and $\|BS\|$ are bounded above by $\cos\beta \sin\theta < \cos\beta \sin\beta$, as $\theta < \beta \leq \pi/2$ by assumption. But this is strictly less than l_2 by Inequality 7. □

Lemma 4 *Consider a petal in a restricted configuration. Throughout any motion during which θ is strictly less than β, the petal remains in a restricted configuration.*

Proof: Consider, in turn, the three conditions required for a restricted configuration. Condition (i) holds throughout the motion by assumption.

In any configuration, B must be on $\mathcal{C}$. Since B starts the motion on the small arc PQ, for condition (ii) to be violated, B must pass through point P or through point Q. P and Q are on the interior of links OA' and OA, respectively, and since $\theta < \beta$, $\mathcal{C}$ always properly intersects these links. Thus B may not move through P or Q, hence condition (ii) holds throughout the motion.

Given that B remains on the small arc PQ, points R and S are well defined. As C starts the motion inside $\square ORBS$, condition (iii) is violated only if C passes through one of the sides of this quadrilateral. Sides OR and OS are portions of links, so C may not pass through them. By Lemma 3, $\|BR\|$ and $\|BS\|$ are both strictly less than $\|BC\|$, so C may not pass through either of these sides. Thus, condition (iii) holds throughout the motion. □

The next three lemmas concern the minimum value for θ attainable by allowed motions from a restricted configuration. Given points X and Y, *ray XY* denotes the ray starting at X, and passing through Y. Let *wedge XOY* designate the closed set of points swept out by a ray starting at ray OX, moving counterclockwise to ray OY. A wedge XOY is only defined if ray $OX \neq$ ray OY, which ensures that the interior of a wedge is non-empty.

Define a *normal* configuration of a petal to be one that has joints B and C in wedge $A'OA$, and no joints other than possibly B and C in the interior of wedge $A'OA$. In other words, joints B and C must be in the wedge $A'OA$, and any other joints of the tree are outside or on the boundary of wedge $A'OA$.

Remark 5 *If all petals are in a restricted configuration, they are also all in a normal configuration.*

Lemma 6 *For a petal in a normal configuration, if neither B nor C lies on ray OA', then θ can be reduced monotonically. The petal remains in a normal configuration.*

Proof: Freeze the link angles at A and B, and rotate the petal as a whole, clockwise about O. If this motion is not possible, one of the three petal links must be blocked by another link. But such a blocking link must intersect the interior of wedge $A'OA$. By assumption, there are no such links, so the motion is possible. The motion monotonically decreases the wedge angle. Closing the wedge in this manner cannot introduce new joints into the wedge interior, so the configuration remains normal for this petal. □

Lemma 7 *For a petal in a normal configuration, if C lies on ray OA' but B does not, then θ can be reduced monotonically. The petal remains in a normal configuration.*

Proof: There are two cases: either C is at O, or it is not. In the first case, we can freeze the link angles, and rotate as in the proof of the previous lemma. In the second case, C has room to swing away from ray OA' (about B) until neither B nor C is on OA'. This does not change θ. Now Lemma 6 applies. As in the previous lemma, the petal wedge only becomes smaller, so no joints are introduced into the wedge interior, and the configuration remains normal for the petal. □

Lemma 8 *The minimum wedge angle of a normal and restricted petal is θ_m.*

Proof: We prove that the minimum wedge angle of a normal and restricted petal occurs in the configuration where C is at O, and $B \in OA'$. Suppose, for contradiction, this is false, and consider a configuration with minimal wedge angle θ.

If B is not on ray OA', then by Lemmas 6 or 7, we can reduce θ monotonically, leaving the petal in a normal configuration. The motion reducing θ also leaves the petal in a restricted configuration. Indeed, $\theta < \beta$ at the start of the motion, θ decreases during the motion, and thus Lemma 4 applies. Hence B must lie on ray OA'. Since the configuration is restricted, B is actually at point P and thus $B \in OA'$.

Since the configuration is restricted, we know that C is inside $\Box ORBS$ (which is $\triangle OBS$, since B is at R). It follows that C may be rotated about B on to OA', without changing θ. Now, we have $BC \subset OA'$, with C not at O. Clearly, sliding C towards O will decrease θ. Again, such motions leave the petal in a normal and restricted configuration.

Thus we can find a restricted configuration with a smaller θ value, contradicting the choice of initial configuration. The result follows by the definition of θ_m. □

6 Trees Can Lock

This section describes our main result: a realizable tree linkage in a simple configuration that cannot be straightened.

From Equation 8, we have $\beta = n\bar{\theta} - (n-1)\alpha$, using $2\pi = n\bar{\theta}$. Hence, $\beta - \bar{\theta} = (n-1)(\bar{\theta} - \alpha) > 0$, by Inequalities 1, 12 and 11. Using these inequalities once more, it follows that

$$\alpha < \theta_m < \bar{\theta} < \beta, \tag{15}$$

hence (α, β) is a non-empty interval. For the $n = 12$ example mentioned in Lemma 1, these inequalities become $29.3° < 29.5° < 30° < 37.6°$.

Consider a tree in a configuration in which the petal configurations are all congruent, with petal angles all equal to $\bar{\theta} \in (\alpha, \beta)$. We place joint B on the small arc PQ. Because $l_2 < l_P(\bar{\theta}) = \|OP\|$ (Inequality 6) we can place B near, but not at, P to ensure $\|OB\| > \|BC\|$. This ensures that joint C may be placed along OB, which is inside $\square ORBS$. The configuration is simple. Furthermore, it satisfies all the conditions of a restricted configuration. Yet it cannot be straightened, as the next theorem shows.

Theorem 9 *Consider a tree of n petals in a configuration such that $\theta_i \in (\alpha, \beta)$ for $0 \leq i < n$, and with each petal in a restricted configuration. Such a configuration can not be straightened.*

Proof: Suppose, to the contrary, that the configuration can be straightened. Choose joint O as the root of the tree. Since all the joints A_i are children of O, they coincide when the tree is straightened. Hence, $n - 1$ petal angles go to zero (the last goes to 2π).

For the putative motion that straightens the configuration, let t_α be the first instant that some petal angle, say for petal k, reaches α. Let t_β be the first instant that some petal angle reaches β.

Note that if $t_\beta < t_\alpha$, then at time t_β all angles are strictly greater than α and at least one is equal to β. This means

$$2\pi = \sum_{i=0}^{n-1} \theta_i > (n-1)\alpha + \beta = 2\pi,$$

a contradiction. Hence $t_\alpha \leq t_\beta$.

During the straightening motion, the joint angles change continuously in time. Since $\alpha < \theta_m$ by Inequality 12, and θ_k approaches α from above as t approaches t_α from below, we may choose $t_0 < t_\alpha$ such that $\theta_k \in (\alpha, \theta_m)$ at time t_0.

Note that during the motion up to time t_0, all petal angles are strictly less than β, as $t_0 < t_\alpha \leq t_\beta$. By Lemma 4 all petals remain in a restricted configuration throughout the interval $[0, t_0]$. By Remark 5, all the petals are also in a normal configuration, hence Lemma 8 applies to petal k. This means $\theta_k \geq \theta_m$, contradicting the choice of t_0.

Thus no motion to straighten this configuration exists. $\square$

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References

[DD97] Erik D. Demaine and Martin L. Demaine. Computing extreme origami bases. Technical Report CS-97-22, Department of Computer Science, University of Waterloo, Waterloo, Ontario, Canada, May 1997.

[ELR+98] H. Everett, S. Lazard, S. Robbins, H. Schröder, and S. Whitesides. Convexifying star-shaped polygons. In *Proceedings of the Tenth Canadian Conference on Computational Geometry*. McGill University, Montreal, Quebec, Canada, August 1998.

[HJW84] J. Hopcroft, D. Joseph, and S. Whitesides. Movement problems for 2-dimensional linkages. *SIAM J. Comput.*, 13:610–629, 1984.

[HJW85] J. Hopcroft, D. Joseph, and S. Whitesides. On the movement of robot arms in 2-dimensional bounded regions. *SIAM J. Comput.*, 14(2):315–333, 1985.

[Kan92] V. Kantabutra. Motions of a short-linked robot arm in a square. *Discrete Comput. Geom.*, 7:69–76, 1992.

[Kan97] V. Kantabutra. Reaching a point with an unanchored robot arm in a square. *International Journal of Computational Geometry and Applications*, 7(6):539–550, 1997.

[KK86] V. Kantabutra and S. R. Kosaraju. New algorithms for multilink robot arms. *Journal of Computer and System Sciences*, 32:136–153, 1986.

[Kor85] J. U. Korein. *A Geometric Investigation of Reach*. ACM distinguished disserations series. MIT Press, Cambridge MA, USA, 1985.

[LW95] W. J. Lenhart and S. H. Whitesides. Reconfiguring closed polygonal chains in Euclidean d-space. *Discrete Comput. Geom.*, 13:123–140, 1995.

[Pei96] Naixun Pei. *On the Reconfigurations and Reachability of Chains*. PhD thesis, McGill University, 1996.

[PW96a] N. Pei and S. Whitesides. On the reachable regions of chains. In *Proceedings of the Eighth Canadian Conference on Computational Geometry*, pages 161–166. Carleton University, Ottawa, Ontario, Canada, August 1996.

[PW96b] N. Pei and S. Whitesides. On the reconfiguration of chains. In Jin-Yi Cai and C.K. Wong, editors, *Computing and Combinatorics, Proceedings of the Second Annual International Conference, COCOON '96*, volume 1090 of *Lecture Notes in Computer Science*, pages 381–390. Springer-Verlag, June 1996.

[PW97] Naixun Pei and Sue Whitesides. On folding rulers in regular polygons. In *Proceedings of the Ninth Canadian Conference on Computational Geometry*, pages 11–16. Queen's University, Kingston, Ontario, Canada, August 1997.

[SS83] J. T. Schwartz and Micha Sharir. On the "piano movers" problem II: general techniques for computing topological properties of real algebraic manifolds. *Advances in Applied Mathematics*, 4:298–351, 1983.

[SY96] I. Suzuki and M. Yamashita. Designing multi-link robot arms in a convex polygon. *International Journal of Computational Geometry and Applications*, 6:461–486, 1996.

[vKSW96] M. van Kreveld, J. Snoeyink, and S. Whitesides. Folding rulers inside triangles. *Discrete Comput. Geom.*, 15:265–285, 1996.

[Whi92] Sue H. Whitesides. Algorithmic issues in the geometry of planar linkage movement. *The Australian Computer Journal*, pages 42–50, May 1992.