Locked Thick Chains

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Abstract

We investigate when *thick* 3D polygonal chains are *locked*, i.e., have a disconnected configuration space. In particular, we show that thick 4-chains are never locked, and we exhibit a class of locked thick 5-chains whose ratio of maximum edge length to minimum edge length is strictly less than 3 (the best known ratio for nonthick chains).

1 Introduction

In the study of linkage folding (see, e.g., [5, Part I]), the typical mathematical model of a mechanical linkage is a collection of rigid line segments (*bars*) permanently attached at certain endpoints (*joints*). Although there are many types of joints, the most common mathematical model is a *universal* joint, which can flex arbitrarily. Bars are usually considered physical objects that cannot intersect each other (usually in 3D). Thus a linkage can move according to any continuous motion such that the bar lengths remain preserved, the bars remain attached according to the joints, and the bars do not intersect each other.

In this paper, we consider changing this model to allow the bars to have positive thickness. Our motivation is that such a model is physically more realistic, in particular when modeling physical objects such as mechanical linkages or protein backbones. O'Rourke [5, Sect. 6.3.3, p. 91] introduced one model of thick 3D linkages, where the chain is a Minkowski sum of a regular nonthick 3D linkage and a ball, turning edges into cylinders and vertices into identical spheres. We distinguish the thick edges and thick joints of the resulting shape from the original nonthick bars and nonthick joints. The linkage is considered *non-self-intersecting* if the only thick bars that intersect each other are those whose underlying nonflat edges share a nonthick vertex. A slight variation is easier to build in practice, and even exists in many magnetic ball/rod construction kits: if two thick edges intersect, they should be from incident nonthick bars *and* the thick edges should intersect only within their shared thick joint. So far no nontrivial theorems have been established in either model.

This paper considers thick "locked chains" in O'Rourke's model and our variation thereof. A *chain* is a linkage in which the bars form a polygonal chain, connected in a path configuration. A linkage is *locked* if its configuration is disconnected, that is, there are two configurations that cannot reach each other by continuous (non-self-intersecting) motions.

O'Rourke originally introduced thick linkages with the following open problem: is there a locked thick chain whose underlying nonthick bars have unit length? [5, Open Prob. 6.1, p. 91] This problem remains open. It is motivated by the analogous problem for nonthick chains [1, 5, Open Prob. 9.19, p. 151], which also remains open. A more general version of the problem was posed by Demaine and O'Rourke [5, Open Prob. 9.20, p. 152]: what is the smallest possible length ratio between the longest bar and the shortest bar that admits a locked nonthick chain? The smallest ratio known is the original five-bar chain which achieves a ratio of $3+\varepsilon$ [2, 1, 5, Thm. 6.3.1, p. 89]. The case of unit-length bars is the smallest possible ratio of 1. But so far nothing better than 3 is known, for nonthick or thick chains.

We analyze a family of locked thick chains that achieve a length ratio of strictly less than 3. In fact, the family is parameterized by the bar thickness as a multiple λ of the minimum bar length. Whenever a chain with the necessary structural properties exists, we prove that the chain is locked when its bar-length ratio is at least $\sqrt{9 - 8\cdot\left(\frac{\lambda}{1-\lambda}\right)^2}$, which is strictly less than 3 for all $\lambda > 0$.

Our locked thick chain has five bars, the same as the classic example of a locked nonthick chain. Indeed, it is known that all locked nonthick chains have at least five bars [2]. As a complementary result, we prove that all locked thick chains have at least five bars as well.

2 Preliminaries

For both thick and non-thick chains, we distinguish the chain (sequence of bars moving in space) from its configuration (a chain in a particular position). A polygonal chain P in $\mathbb{R}^d$ is a sequence of fixed length bars connected at their successive endpoints and moving freely in a d-dimensional space. The chain has $n+1$ vertices $V = \langle v_0, \ldots, v_n\rangle$, and is specified by the fixed edge lengths d_i between v_i and v_{i+1}, $i = 0, \ldots, n-1$. We write $P[i,j]$, $i \le j$, for the polygonal subchain composed of vertices $v_i, \ldots, v_j$. The *joints* correspond to the internal vertices $v_1, \ldots, v_{n-1}$.

A *configuration* $Q = \langle q_0, \ldots, q_n\rangle$ of the chain P is an embedding of P into $\mathbb{R}^d$, i.e., a mapping of each vertex v_i to a point $q_i \in \mathbb{R}^d$, satisfying the constraints that the distance between q_i and q_{i+1} is d_i. The points q_i and q_{i+1} are connected by a straight line segment e_i.

Let B^d_λ be the ball in $\mathbb{R}^d$ of radius λ centered at the origin, and let e be a straight line segment in $\mathbb{R}^d$. The *thick bar* of thickness λ and of skeleton e is the Minkowski sum $e \oplus B^d_\lambda$ between e and B^d_λ. A *thick chain* P_λ in $\mathbb{R}^d$ is

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specified by its skeleton P and its thickness λ. It can be seen as a sequence of thick bars whose end balls coincide. A configuration Q_λ of a thick chain P_λ is the Minkowski sum between a configuration Q of P and the ball B^d_λ. The corresponding configuration Q of P is called the *skeleton* of configuration Q_λ. Note that adjacent thick bars of a thick chain always intersect. We however impose that a configuration be *simple*, that is, nonadjacent bars do not intersect. Furthermore, we require that the thick bar $e_{i-1} \oplus B^d_\lambda$ does not intersect the ball $q_{i+1} \oplus B^d_\lambda$. A motion of a chain is simple if every configuration during the motion is simple.

An *expansive motion* of a chain P is a motion with the property that the distance between any pair of points on the chain monotonically increases with time [3]. We say that the motion of a thick chain is *expansive* if the motion of its skeleton is expansive. We can then show the following:

Lemma 1 *An expansive motion of a thick chain starting from a simple configuration, is simple.*

3 Thick 4-Chains Cannot Lock

It is known that a nonthick 4-chain cannot be locked [2]. Following an idea similar to that in [4], we use a linear transformation as a first step toward bringing the skeleton of the thick chain in 2D. Define the parameterized linear transformation $f_\tau : \mathbb{R}^3 \to \mathbb{R}^3$ by $f_\tau(x, y, z) := (\tau x, y, z)$ with parameter $\tau \geq 1$. For a set $S \subseteq \mathbb{R}^3$, write $f_\tau(S) = \{f_\tau(p) : p \in S\}$. Note that linear functions are distributive over the Minkowski sum: that is, for all sets A and B,

$$f_\tau(A \oplus B) = f_\tau(A) \oplus f_\tau(B). \tag{1}$$

Further, because $\tau \geq 1$, we have

$$B^3_\lambda \subseteq f_\tau(B^3_\lambda). \tag{2}$$

Theorem 2 *Every simple thick 4-chain can be straightened in 3D.*

Proof. Let $C_\lambda = C \oplus B^3_\lambda$ be a thick 4-chain of thickness λ of skeleton C whose vertices are $(v_0, v_1, v_2, v_3, v_4)$.

Consider the plane K formed by the two middle nonthick bars of the skeleton, $e_1 = (v_1, v_2)$ and $e_2 = (v_2, v_3)$. We may choose the coordinate system so that K is the yz plane. We can also apply small perturbations to v_0 and v_4 to ensure that they are not in K. Let k_1 and k_2 be the first and last (nonthick) bars of C, that is, $k_1 = (v_0, v_1)$ and $k_2 = (v_3, v_4)$.

Definition the parameterized linear transformation f_τ as above. The two middle bars, e_1 and e_2, have the same length in $f_\tau(C)$ as in C, but the transformation increases the length of the two end bars, k_1 and k_2. So we have to truncate the length of k_1 and k_2 to preserve their length. Also, the thick bars of C_λ, which are cylinders with a sphere at each end, become cylinders with a ellipsoid basis in $f_\tau(C_\lambda)$. In order to obtain a thick chain again, the basis of this cylinder has to be changed to a disk.

By truncating the length of the end bars, we obtain a new 4-chain C^τ from $f_\tau(C)$. Clearly, $C^\tau \subseteq f_\tau(C)$. By equation (2), $B^3_\lambda \subseteq f(B^3_\lambda)$. So,

$$C^\tau_\lambda = C^\tau \oplus B^3_\lambda \subseteq f_\tau(C) \oplus f_\tau(B^3_\lambda) = f_\tau(C \oplus B^3_\lambda) = f_\tau(C_\lambda).$$

We want to apply the linear motion f_τ for τ ranging from 1 to ∞. To achieve this motion in finite time, we define another motion parameterized by t from 0 to 1 that applies f_τ where $\tau = 1/(1-t)$. Throughout the motion, we truncate the length of the two exterior segments and the basis of the ellipsoid cylinder. Because the linear transformations preserve intersection among regions, two thick bars that do not intersect before the motion do not intersect during the motion. As t approaches 1, τ grows to infinity and the exterior thick bars become perpendicular to the yz plane. Let C' be that skeleton of the chain at that point in time.

Two possible situations arise: either the vertices v_0 and v_4 are on the same side of the yz plane or they lie on opposite sides of the yz plane; see Fig. 1 (a–c) for top and side views. Let Π_1 be the plane containing v_0, v_1, and v_2, and Π_2 be the plane containing v_2, v_3, and v_4. Let ℓ be the line at the intersection of Π_1 and Π_2. Notice that ℓ intersects C' only at vertex v_2. Thus the subchain $v_0v_1v_2$ is contained in a halfplane of Π_1 bounded by ℓ and the subchain $v_2v_3v_4$ is contained in a halfplane of Π_2 bounded by ℓ. By hinging those two halfplanes about ℓ, we obtain an expansive motion of C' that makes it planar. By Lemma 1, this motion is simple for the thick chain.

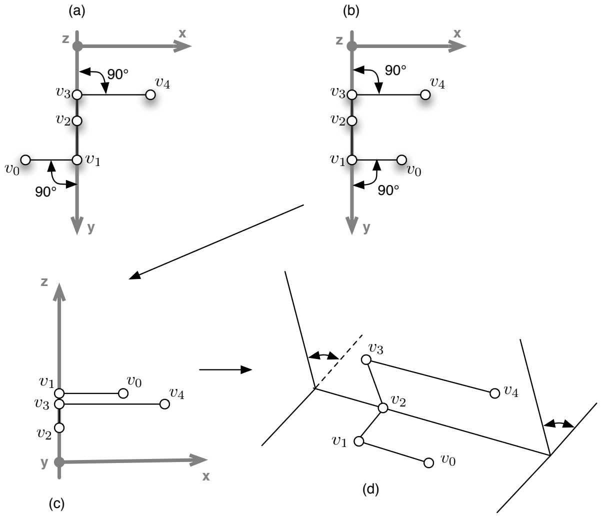

Figure 1: 4-chain: Example where we bring back the two exterior thick bars in the plan yz.

Finally, we straighten the resulting planar 4-chain using an expansive motion [3]. Again, by Lemma 1, this motion is simple for the thick chain. □

4 Locked Thick 5-Chains

4.1 Introduction

Consider a 5-chain of thickness λ with end bars (v_0, v_1) and (v_4, v_5) of length $s > 1$ and middle-bars (v_1, v_2), (v_2, v_3) and (v_3, v_4) of length 1. Because the chain is simple, $\lambda < 1/2$. We will determine bounds on s so that the chain can be locked.

Fig. 2 shows the orthogonal projection of two adjacent thick bars (thickness λ) and one more thick bar perpendicular to the other two. The projection is over to the plane formed by the skeleton of the two adjacent thick bars. We first bound the distance between the points C and v_0 depending on the angle α between the two adjacent thick bars and on the thickness λ.

Lemma 3 *Consider in 2D two adjacent thick bars (v_0, v_1) and (v_0, v_2) of thickness λ, consider a circle of radius λ centered at the point C which is in the interior of the angle α formed by the two adjacent thick bars and which does not intersect any of these two thick bars (see Fig. 2). Let d be the distance between v_0 and C. Then*

$$d \geq \frac{2 \cdot \lambda}{\sin(\alpha/2)}.$$

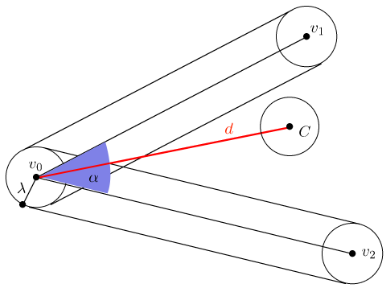

Figure 2: Lemma 3.

Lemma 4 *Let P_λ be a 5-chain of thickness λ (see Fig. 3) which is the Minkowski Sum of a non-thick 5-chain P of vertices $(v_0, v_1, v_2, v_3, v_4, v_5)$ with the ball B^3_λ. If the three middle bars are of length one, and the central projection from v_4 on the plane $v_1v_2v_3$ is as in Fig. 4, then the distance s between the middle point of the three middle thick bars of P and the vertex v_1 (or the vertex v_4) is smaller than*

$$\frac{1}{2}\sqrt{9 - 8 \cdot \left(\frac{\lambda}{1-\lambda}\right)^2}.$$

Proof. Consider the central projection of P_λ from v_4 on plane K containing v_1, v_2, and v_3 (see Fig. 4). The thick bar v_4v_5 intersects K in an ellipsoid centered at the point D. This ellipsoid contains a circle of radius λ centered at D. Thus replacing the ellipsoid by a circle centered at D is a weaker constraint on the motion. Applying Lemma 3, we bound the distance d between v_1 and D:

$$d \geq \frac{2 \cdot \lambda}{\sin(\alpha/2)}$$

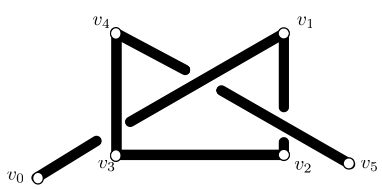

Figure 3: Thick 5-chain.

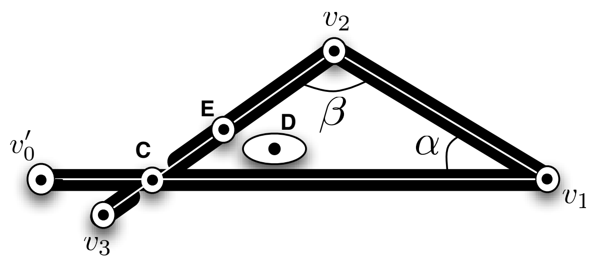

Figure 4: Central projection of the thick 5-chain from v_4 on the plane K containing v_1, v_2, and v_3. The thick bars v_1v_2, v_2v_3 and v_0v_1 are shown, as well as the ellipsoid formed by the thick bar v_4v_5 as it intersects plane K.

Because the projection is on K, the length of thick bars v_1v_2 and v_2v_3 are preserved and their thickness can only be larger. Let v_0' be the projection of v_0 on K and C be the intersection in the projection between $v_0'v_1$ and v_2v_3. By assumption, this intersection always exists. Consider the triangle v_1v_2C. Since $|v_2v_3| = 1$, $|v_2C| \leq 1 - 2\lambda$. Also $|v_1v_2| = 1$ and by the triangle inequality, $d = |v_1C| \leq 2 - 2\lambda$.

Because the central projection is from v_4 over K, $|v_0'v_1| > |v_0v_1|$.

$$2 - 2 \cdot \lambda > d \geq \frac{2 \cdot \lambda}{\sin(\alpha/2)} \tag{3}$$

We have to bound the angle α by the equation (3):

$$\begin{aligned} 2 - 2 \cdot \lambda &> \frac{2 \cdot \lambda}{\sin(\alpha/2)} \\ \sin(\alpha/2) &> \frac{\lambda}{1-\lambda} \\ \frac{\alpha}{2} &> \arcsin \frac{\lambda}{1-\lambda} \\ \alpha &> 2 \arcsin \frac{\lambda}{1-\lambda} \end{aligned}$$

Because, in every triangle, the sum of angles is equal to π, we have β to $\beta < \pi - \alpha$.

With the angles α lower-bounded and β upper-bounded, we have enough information to bound the distance between the middle point of the three middle thick bars of P and the point A (the vertex v_1).

Let be $L = l_2 + l_3 + l_4 = 3$ the total length of the three middle thick bars and E the middle point of the three middle thick bars: the distance between E and the vertices v_4 and v_1 is equal to $3/2$. By the length attributed to the three middle thick bars, the point E is at the center of the thick bar v_2v_3 (see Fig. 4).

Consider the triangle EBA (obtained by considering the point E in Fig. 4) to find the distance between the points E and A.

In this new triangle, we know : the angle β ($\beta < \pi - \alpha$), the length $|AB|$ (by the definition of P, $|AB| = 1$) and the length $|EB|$ (by the definition of P and the middle point E, $|EB| = \frac{1}{2}$).

By the Al-Kashi's theorem, we have $|EA| = \sqrt{|AB|^2 + |EB|^2 - 2 \cdot |AB| \cdot |EB| \cos\beta}$. $|EA|$ corresponds to the distance between the middle point E of the three thick bars of P and the vertex v_1 so $|EA| = \sqrt{1 + \frac{1}{4} - 2 \cdot 1 \cdot \frac{1}{2}\cos\beta}$. This distance is the same if we consider the vertex v_4 instead of the vertex v_1 because the three middle thick bars have length one.

So $s = |EA| = \sqrt{1.25 - \cos\beta}$:

$$
\begin{array}{lrcl}
\text{As} & \beta & < & \pi - \alpha \\
\text{So} & \cos(\beta) & > & \cos(\pi - \alpha) = -\cos\alpha \\
\text{And so} & s & < & \sqrt{1.25 + \cos\alpha} \\
\text{As} & \alpha & > & 2\arcsin\frac{\lambda}{1-\lambda} \\
\text{Then} & \cos\alpha & < & \cos(2\arcsin\frac{\lambda}{1-\lambda}) \\
\text{But} & \cos(2\arcsin\frac{\lambda}{1-\lambda}) & = & \cos^2\arcsin\frac{\lambda}{1-\lambda} - \sin^2\arcsin\frac{\lambda}{1-\lambda} \\
 & & = & 1 - 2\sin^2\arcsin\frac{\lambda}{1-\lambda} \\
 & & = & 1 - 2\left(\frac{\lambda}{1-\lambda}\right)^2 \\
\text{Then} & \cos\alpha & < & 1 - 2(\frac{\lambda}{1-\lambda})^2 \\
\text{And so} & s & < & \sqrt{1.25 + 1 - 2\left(\frac{\lambda}{1-\lambda}\right)^2} \\
 & & < & \sqrt{2.25 - 2(\frac{\lambda}{1-\lambda})^2}
\end{array}
$$

□

This lemma bounds distances from the middle point E of the three middle bars of the thick 5-chain. We will re-use the demonstration of [5] but with this distances as radius of the centered sphere.

Theorem 5 *Let K be a 5-chain of thickness λ (see Fig. 3), result of the Minkowski Sum between a non-thick 5-chain of vertices $(v_0, v_1, v_2, v_3, v_4, v_5)$ and the ball B^3_λ. If the length of each of the three middle thick bars is set to one and the end bars are of length greater than $\sqrt{9 - 8(\frac{\lambda}{1-\lambda})^2}$ then K is locked.*

Proof. Let $D = \frac{1}{2}\sqrt{9 - 8 \cdot (\frac{\lambda}{1-\lambda})^2}$ be the upper-bound on the distance between the middle point E of the three middle thick bars of K and the vertex v_1 (or v_4) given by the lemma 4. Let $r = D + \varepsilon$, for $\varepsilon > 0$ small and the sphere F of radius r centered at E. By construction, the vertices $\{v_1, v_2, v_3, v_4\}$ are in F for all reconfiguration of K.

We fix l_1 and l_5 to at least $2 \cdot r + \varepsilon = 2 \cdot D + 3 \cdot \varepsilon$.

Because l_1 and l_5 are greater than the diameter of F, the vertices v_0 et v_5 are not in F for all reconfiguration of K.

We first claim that unless v_0 or v_5 enter F, the projection from v_4 on the plane through $v_1v_2v_3$ is as in Fig. 4, that is, the projection of v_0v_1 intersects v_2v_3. Symmetrically, we claim that in the projection from v_1 on the plane containing $v_2v_3v_4$, the projection of v_4v_5 intersects v_2v_3. Suppose that the former claim gets violated first during the motion. In that case, the projection of v_0 has to lie on v_2v_3 but in that case, v_0 is inside the triangle $v_2v_3v_4$ which itself is inside F, a contradiction.

Assume by contradiction that there is an unlocking motion for K.

If we close K by adding a string along the surface of F between its two free ends, then we obtain a trefoil knot.

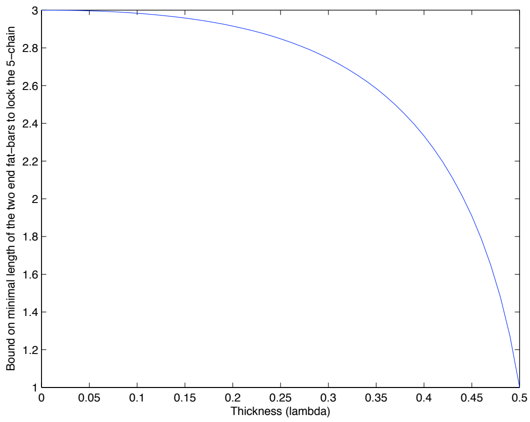

Figure 5: Bounds from Theorem 5

Because F separates (by its boundary) the two sets $\{v_0, v_5\}$ and $\{v_1, v_2, v_3, v_4\}$, we can attach a sufficiently long unknotted string s from v_0 to v_5 exterior to F. We obtain a trefoil knot.

By our assumption, we can, by an existing motion, unlock K (note that the topology of a knot does not change during a motion), at the end of this motion, we obtain a unlocked knot (the trivial knot). This result is in contradiction with the fact that $K \cup s$ is a trefoil knot when we add to it a string s, then K cannot be unlock by any motion. □

We have first bound on the minimal length of the two end bars needed to lock a thick 5-chain of thickness λ. A first plot may be drawn using this formula (see Fig. 5). In this plot, the x axis represent the thickness (λ) and the y axis our bound on the minimal length to assign to the two end bars of the thick 5-chain.

Interestingly, if we put the thickness parameter λ to zero (we are so in the case of a non-thick 5-chain) then we obtain the same result than [5].

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