Meshes preserving minimum feature size

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Abstract. The *minimum feature size* of a planar straight-line graph is the minimum distance between a vertex and a nonincident edge. When a polygon is partitioned into a mesh, the *degradation* is the ratio of original to final minimum feature size. We show that some planar straight-line graphs cannot be triangulated with constant degradation, even with an unbounded number of Steiner points and triangles. This result answers a 14-year-old open problem by Bern, Dobkin, and Eppstein. For an n-vertex input, we obtain matching worst-case lower and upper bounds on degradation of $\Theta(\lg n)$. Our upper bound comes from a new meshing algorithm that uses $\mathcal{O}(n)$ triangles and $\mathcal{O}(n)$ Steiner points. If we allow triangles to have Steiner points along their sides, a construction is presented that achieves $\mathcal{O}(1)$ degradation.

Introduction

In this paper[1], we study the problem of polygon triangulation, with the possible aid of Steiner vertices. Our goal is to not introduce small distances between vertices and non-incident edges, compared to distances already existing in the shape. To compare the input and output, we use the *minimum feature size* of a planar straight-line graph G, denoted by $\mathrm{mfs}(G)$. This is the minimum distance between a vertex and a nonincident edge. We are interested in decomposing a polygon P into a planar straight-line graph (more specifically, a triangulation) G such that the minimum feature size of G is as close as possible to that of P. We call the ratio $\frac{\mathrm{mfs}(P)}{\mathrm{mfs}(G)}$ the *degradation* of the decomposition of P into G. Note that mfs does not distinguish between the interior and exterior of P when measuring distances.

Minimum feature size is a parameter well suited for describing the resolution needed to visually distinguish elements in a mesh. For example, it measures the maximum thickness that the edges in a mesh can be drawn. Also, mfs measures the amount of error allowed in the placement of vertices, so that a drawing preserves its topology. This could be useful in manufacturing, as well as in finite element simulation.

One important issue here is the type of desired triangulation. This choice has a large effect on the results that can be achieved. See Figure 1. The most common decomposition of a polygon is the *classic triangulation*, where noncrossing chords are added between vertices of P, until the interior of P is partitioned into triangles. If we allow Steiner points, a *proper triangulation* is

[1]The research presented in this paper was initiated at the 24th Annual Winter Workshop on Computational Geometry at the Bellairs Research Institute of McGill University, at which the presence of Ferran Hurtado was appreciated and is hereby acknowledged.

such that any two edges that lie on the same interior face and are incident to a common vertex are not collinear. A *nonproper triangulation* simply partitions P into triangles, with no restrictions.

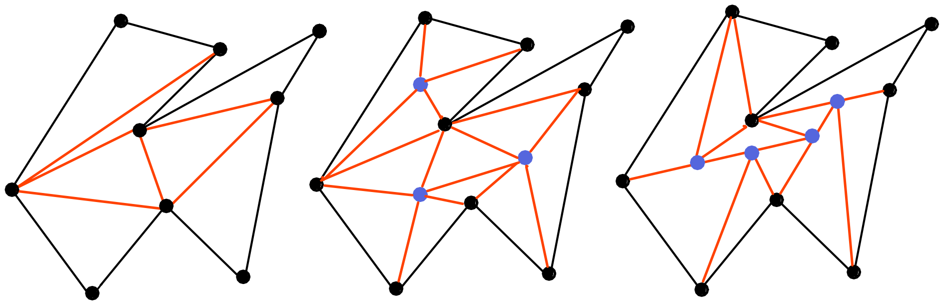

FIGURE 1. Types of triangulations: classic, proper, nonproper. Steiner points are blue.

Bern, Dobkin, and Eppstein [**BDE95**] studied this problem, using the notion of *internal feature size* $\mathrm{ifs}(P)$, which is the minimum distance inside P between a vertex and a nonincident edge[2]. They proved that every polygon P (possibly with holes) has a nonproper triangulation in which every triangle has height $\Omega(\mathrm{ifs}(P))$.

For a planar straight-line graph, triangulating all of its faces with triangles that have height at least h is equivalent to guaranteeing that the triangulation itself has minimum internal feature size at least h. Notice that the internal feature size of a triangle equals its height. Thus $\mathrm{ifs}(P)$ is a lower bound on the smallest height of a triangle in any triangulation of P, so this bound is the best possible up to constant factors. However the method does not guarantee that the minimum feature size of the resulting triangulation is bounded by a function of $\mathrm{mfs}(P)$. Thus, partitioning both the inside and the outside of a polygon into triangles whose height is bounded by a function of $\mathrm{mfs}(P)$ is not guaranteed either. Consequently, the first open problem the authors list is whether their result can be generalized to planar straight-line graphs, that is, whether such graphs can be triangulated while preserving their minimum feature size.

We answer this open problem negatively. Specifically, we provide a simple polygon G such that every proper triangulation of G has degradation $\Omega(\lg n)$, independent of the number of Steiner points and triangles.

We match this lower bound by providing an algorithm for properly triangulating any given planar straight-line graph G so that degradation is $\mathcal{O}(\lg n)$. Our algorithm uses $\mathcal{O}(n)$ Steiner points and hence $\mathcal{O}(n)$ triangles. Steiner points are necessary to obtain a degradation smaller than a linear factor; Bern and Eppstein [**BE95**] showed that all classic triangulations of a regular n-gon have a minimum feature size degradation of $\Omega(n)$. This can be extended trivially to quadrangles or any decomposition with constant size faces.

Until now, no meshing algorithm with a constant degradation was known. Ruppert's Delaunay mesh refinement algorithm claims such a bound [**Rup93**, Theorem 1],but the constant actually depends on the minimum angle of the input graph (as well as the minimum triangle angle guaranteed by the algorithm).

What causes the need for logarithmic degradation in proper triangulations of planar straight-line graphs? We show that the essential issue is forbidding Steiner points along the sides of a triangle. However, by allowing Steiner points along the sides of triangular elements (what we call *nonproper triangulation*), $\mathcal{O}(1)$ degradation is actually possible.

[2]Note that "internal feature size" is called "minimum feature size" in [**BDE95**].

1 Nonproper Triangulations can Preserve Minimum Feature Size

In this section we show how to construct a nonproper triangulation for any polygon P, such that the minimum feature size degradation is $\Theta(1)$. We use $\Theta(n)$ Steiner points, and the construction can be computed in linear time.

We provide a brief overview of our construction. There are two distinct regions of P which will be triangulated separately. The two regions will be separated by a polygon Q interior to P whose boundary remains at distance $\Theta(\mathrm{mfs})$ to that of P. The region between P and Q is called the *tube*. The algorithm places all Steiner points on or interior to Q; none are placed on P.

The polygon Q is constructed to have the following properties with respect to absolute constants $c_1, c_2, \ldots c_3$: (1) All points on Q are at most c_1 away from the closest point on P, and at least c_2 away from the nearest point on P. (2) All vertices on Q have y-coordinates which are multiples of c_3 (i.e., they are on a c_3 horizontal grid). (3) There are $\mathcal{O}(n)$ vertices (initially) on Q. The details of how to find such a Q are omitted, but we note that the *grassfire* transformation plays a vital part in the construction.

Next, an edge from each vertex of Q is added to the closest vertex on P; now the region between the P and Q is subdivided into triangles and quadrangles. The interior of Q is then quadrangulated by performing a trapezoidal decomposition of the interior of the tube; this will introduce new Steiner vertices on Q. However, since all vertices on Q are at least c_3 separated, this does not cause any issues with respect to minimum feature size.

The decomposition at this point contains triangles and quadrangles; the quadrangles need to be triangulated. The most difficult cases are the quadrangles in the tube; these will have one edge from P and one edge from Q. The edge from Q may have many Steiner vertices introduced by the trapezoidal decomposition, separated by a distance of at least c_3. The edge from P does not have any Steiner vertices, and it can not be assumed that it is safe to put any Steiner vertices on P because there could be another edge of P arbitrary close. For example see Figure 2 (D) where a constant faction of the boundary of P is off-limits to the introduction of Steiner points). The other two edges connecting P and Q do not have Steiner points. Figure 2 (A) shows how to triangulate a rectangle subject to these restrictions while maintaining constant minimum feature size; this construction can be slightly modified to decompose any of the needed quadrangles and complete the construction, yielding the following theorem.

Theorem 1.1 *Every polygon has a non-proper triangulation with constant minimum feature size degradation.*

2 Degradation Upper Bound for Proper Triangulations

The decomposition method is the same as for non-proper triangulations, with the exception that the decomposition of quadrangles shown in Figure 2 (A) can not be used as it is a non-proper triangulation. Instead the construction at the bottom of Figure 2 (C) is used; this yields $\mathcal{O}(\log n)$ degradation.

Theorem 2.1 *Every n-vertex polygon has a proper triangulation with $\mathcal{O}(\log n)$ minimum feature size degradation.*

3 Lower Bounds

Due to lack of space, and complexity of proofs, we only state our main results of this section.

Let P_n be the generalized version of the $n + 6$ vertex polygon illustrated in Figure 2 (D). This polygon has width 2, height $n + 1$, and minimum feature size of 1. Let R_n be the rectangular region of P_n shaded in Figure 2 (D).

A *τ-grid* is a set of τ vertical lines. Let a *τ-grid triangulation* of P_n be a nonproper triangulation of P_n where all Steiner vertices are on a τ-grid.

Theorem 3.1 *For every τ and n, every τ-grid (nonproper) triangulation G of P_n has degradation $\Omega(\min\{\frac{\lg n}{\lg \lg n}, \frac{\lg n}{\lg \tau}\})$.*

When considering proper triangulations, our (omitted) proof for Theorem 3.1 simplifies to a $\Omega(\frac{\lg n}{\lg \lg n})$ bound on degradation. However, we are able to improve to the following.

Theorem 3.2 *For every n, every proper triangulation G of P_n has degradation $\Omega(\lg n)$.*

This can be shown to extend to a bound of $\Omega(\lg_r n)$ for proper r-rangulations.

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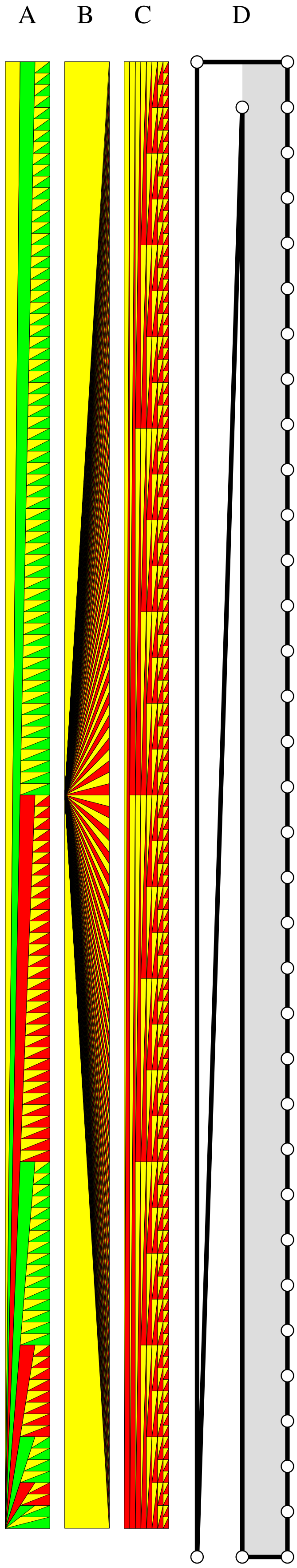

FIGURE 2. (A)-(C) are triangulations of the same rectangle with 129 vertexes on the right edge. (A) is a non-proper triangulation, with $\mathcal{O}(1)$ degradation. For comparison, (B) is a simple proper fan triangulation with $\mathcal{O}(n)$ degradation. (C) is a proper triangulation with $\mathcal{O}(\log n)$ degradation. Observe how for (A) the interiors of all the triangles are clearly visible; in (C) it is more difficult to discern the individual triangles, and in (B) it is impossible. (D) is a polygon that shows Steiner vertices cannot be placed on a significant fraction of the boundary close to the reflex vertex.