Balloon Polyhedra

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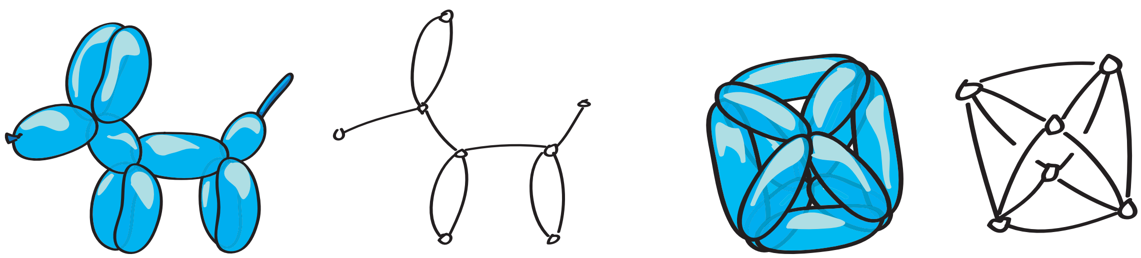
1 Introduction

You have probably seen a long balloon twisted into a dog like Figure 1(a). But did you know that the same balloon can be twisted into a regular octahedron like Figure 1(b)?



(a) Classic dog (one balloon). (b) Octahedron (one balloon).

Figure 1: Two one-balloon constructions and their associated graphs.

Balloon twisting offers a great platform for making, exploring, and learning about both polyhedra and graphs. Even the classic balloon dog can be thought of as a graph, with edges corresponding to balloon segments between the twists. Children of all ages can thus enjoy the physicality of the balloon medium while learning about mathematics.

Here we give a practical guide to making polyhedra and related geometric constructions with balloons, while briefly describing the mathematics and computer science related to balloon construction. The latter is the focus of a companion paper [DDH08], which contains further details.

2 If Euler Were a Clown

What polyhedra can be made by twisting a single balloon, like the octahedron in Figure 1(b)? The balloon must traverse every edge of the polyhedron in sequence, and can traverse each edge only once, though it can (and will) visit each vertex multiple times. Mathematicians will recognize this structure as an *Eulerian path*, as studied by Euler in 1736 [Eul36]. We review this classic mathematics of this structure in the context of balloons by wondering: what is the simplest polyhedron that can be made from a single balloon?

The simplest (nonflat) polyhedron is a tetrahedron, but a tetrahedron cannot be made from a single balloon. Such a balloon would start at some vertex and end at some other vertex, but

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for every remaining vertex, each time the balloon enters the vertex it also exits the vertex. Thus, for a balloon twisting of a graph to possibly exist, every vertex except possibly two (the starting and ending vertices) must have an even number of edges incident to it, or even *degree*. But in the tetrahedron, all four vertices have odd degree. Thus the tetrahedron cannot be made from one balloon.

The four-dimensional analog of a tetrahedron (called the 4-*simplex*) consists of four tetrahedra glued together face-to-face. One 3D projection of a 4-simplex is a regular tetrahedron with all four vertices joined to a fifth vertex added in the center. Every vertex is thus joined to all four other vertices, giving it even degree. So the 4-simplex does not have the "too many odd-degree vertices" obstruction that the tetrahedron had. Indeed, Figure 2 shows a 4-simplex made from one balloon. Building one is somewhat difficult for practical reasons; we suggest you try it, and if you get stuck, see [Mor07].

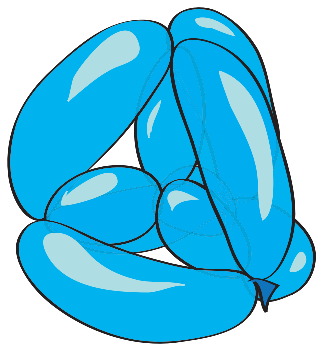

Figure 2: The 4-simplex, made from one balloon, is a puzzle to twist.

In fact, a single balloon can be twisted into any connected graph in which all vertices have even degree. The starting point is to traverse the graph naïvely: start the balloon at any vertex, route it along any incident edge, and keep going, at each step following any edge not already visited. Because the vertex degrees are all even, whenever the balloon enters a vertex, there is always another unvisited edge along which it can exit. The only exception is the starting vertex, where the balloon's initial exit left an odd number of unvisited edges. This vertex is the only place where the balloon might have to stop, and eventually this must happen because the balloon will run out of edges to visit. When this happens, the balloon forms a loop that visits some edges once, but possibly does not visit some edges at all. Now consider just the graph of unvisited edges. It too has even degree at every vertex, because the balloon exited every vertex it entered, including the start vertex. Hence we can follow the algorithm again with a second balloon, and a third balloon, and so on, until the balloons cover all the polyhedron's edges (and no two balloons cover the same edge). Now we take any two balloons that visit the same vertex and merge them into one balloon by the simple switch shown in Figure 3. This also forms a loop, and visits all the same edges. Repeating this process, we end up with one balloon forming a loop that visits every edge exactly once.

Now we have a construction for twisting many graphs from a single balloon. We have seen one 4D polyhedron to which this construction applies, but what about 3D polyhedra? One example is the octahedron from Figure 1(b). The octahedron has six vertices, each of even degree 4. Therefore the general construction applies; Figure 4 shows a practical construction; see also [Sha07, EKR00, Mor07]. The octahedron is actually the only Platonic solid twistable from a single balloon: for all the others, every vertex has odd degree (3 or 5).

Are there simpler 3D polyhedra than the octahedron that are twistable from one balloon? If we

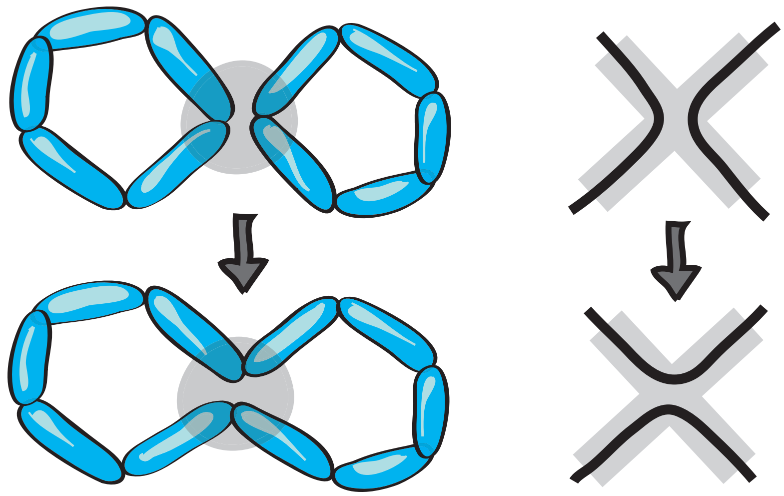

Figure 3: Joining two balloon loops (top left) into one balloon loop (bottom right) when the two loops visit a common vertex (shaded, and abstracted on the right).

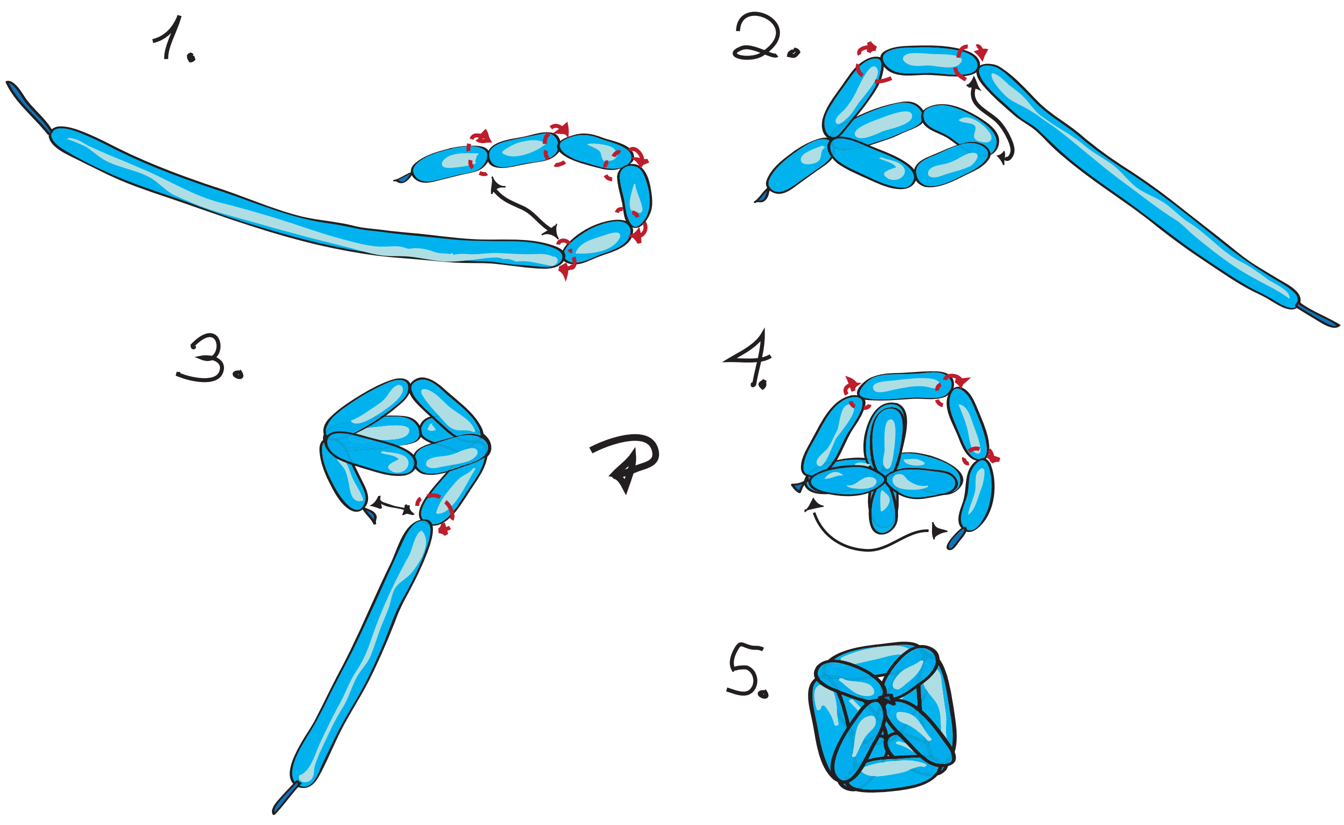

Figure 4: The octahedron is like the balloon dog of balloon polyhedra. With practice, it can be twisted quickly from one balloon.

glue two tetrahedra together, we get a *triangular dipyramid*. The two apices have odd degree 3, but the remaining vertices all have even degree 4. We know that a single balloon cannot tolerate more than two odd-degree vertices, but this polyhedron has just two, putting it right at the borderline of feasibility. Figure 5 shows that it indeed can be made from one balloon,

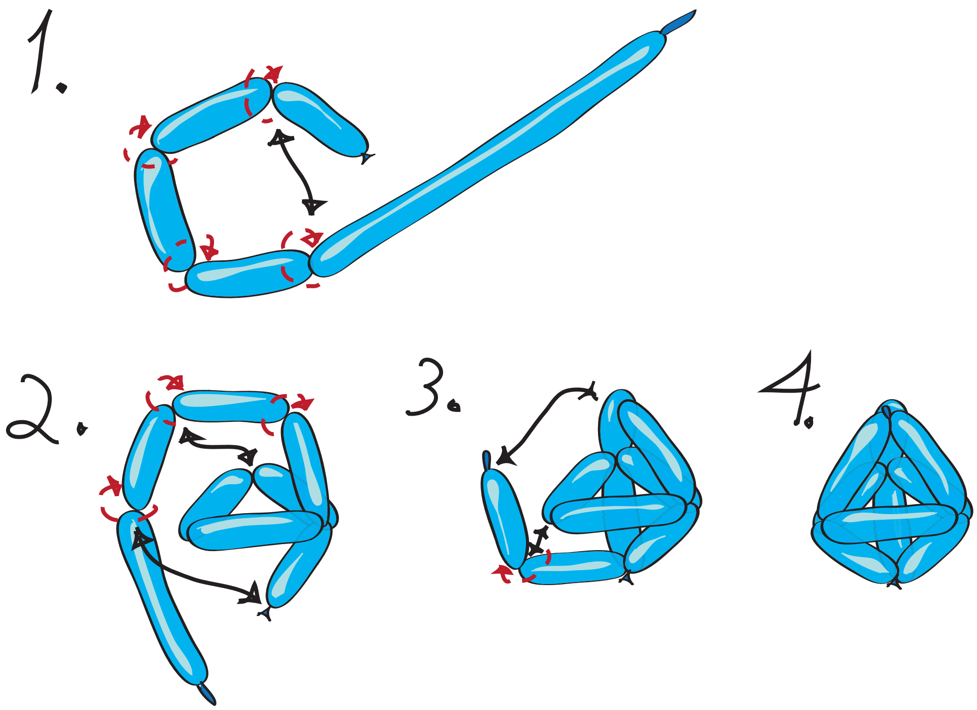

Figure 5: The triangular dipyramid is perhaps the simplest polyhedron twistable from one balloon, and is easier than the octahedron.

More generally, a single balloon can be twisted into any connected graph with exactly two vertices of odd degree. Just imagine adding an extra edge to the graph, connecting the two odd-degree vertices. This addition changes the degrees of the two odd-degree vertices by 1, making them even, and does not change the degree of any other vertices. Hence this modified graph has all vertices of even degree, so it can be twisted from a single balloon forming a loop. We can shift the loop of the balloon so that it starts and ends at one of the odd-degree vertices. Then we remove the added edge, from both the graph and the balloon. We are left with a single balloon visiting every edge of the original graph exactly once. Naturally, the balloon starts at one odd-degree vertex and ends at the other.

Summarizing what we know, a one-balloon graph has at most two odd-degree vertices, and every connected graph with zero or two odd-degree vertices can indeed be twisted from one balloon. What about graphs with one odd-degree vertex? Don't worry about them; they don't exist. Euler [Eul36] showed that every graph has an even number of odd-degree vertices. To see why, imagine summing up the degrees of all the vertices. We can think of this sum as counting the edges of the graph, except that each edge gets counted twice, once from the vertex on either end. Therefore the sum is exactly twice the number of edges, which is an even number. The number of odd terms in the sum (the number of odd-degree vertices) must thus be even.

We conclude that we know all one-balloon graphs. In fact, we have just rediscovered the classic characterization of Eulerian paths, common in graph-theory textbooks, but in the context of balloons. (For example, Bondy and Murty's book [BM76] uses the same trick for the case of two odd-degree vertices.)

If you are looking for a good challenge for making a polyhedron from one balloon, we recommend the cuboctahedron. Every vertex has degree 4, but there are twenty-four edges and sharper dihedral angles, making it difficult to twist except from especially narrow balloons. If you get stuck, see [Mor07], which also has other interesting examples such as the pentagonal antiprism.

3 Cheating with One Balloon

One trick for transforming graphs into one-balloon graphs is to double every edge: whenever two vertices are connected by an edge, add a second edge alongside it. This change doubles the degree of every vertex, so all resulting vertices have even degree. Therefore we can make the doubled graph from one balloon. In other words, one balloon visits every edge in the original graph exactly twice instead of once. The results can look attractive; see, e.g., [Sab08, Mor07].

In fact, we do not need to double every edge: we just need to double edges along paths connecting the odd-degree vertices in pairs, except for one pair of odd-degree vertices which we can leave alone. Minimizing the number of edges we double is the *Chinese postman problem*, a well-studied problem in computer science. Balloons offer a fun context for studying efficient algorithms for this problem.

After we have made a balloon with some of the edges doubled, we could *pop* the extra edges using a sharp object. The result is a more uniform aesthetic. For example, Figure 6 shows how a tetrahedron, which would normally require two balloons, can be made from one balloon with one popped segment. (We leave the construction of the tetrahedron with a doubled segment as an exercise for you. Remember to start and end at the two odd-degree vertices.) In practice, be careful to twist the ends of a segment extensively before popping it, to prevent affecting the incident segments.

Figure 6: Pop-twisting a tetrahedron from one balloon.

On the topic of balloon popping, a challenging type of puzzle is this: given an already-twisted balloon polyhedron, can you pop some of the balloon segments to make another desired graph? This problem is known as *subgraph isomorphism*, and is among the family of computationally intractable "NP-complete" problems [GJ79], so there is likely no good algorithm to solve it.

4 Polyballoon Constructions

If we cannot make a graph with just one balloon, how many do we need? The minimum number of balloons that can make a particular graph, with each edge covered by exactly one segment, is the graph's *bloon number* [DDH08].

There turns out to be a very simple formula for the bloon number: it is half the number of odd vertices [DDH08, Theorem 2].[1]

[1]Unless, of course, the graph has no odd vertices; then the bloon number is 1, not 0.

On the one hand, we cannot hope for fewer balloons: each odd-degree vertex must be the start or end of some balloon, so each balloon can "satisfy" only two odd-degree vertices. On the other hand, there is a construction with just this many balloons, using a simple construction similar to the arguments above. First we add edges to the graph, connecting odd-degree vertices in disjoint pairs. The number of added edges is half the number of odd-degree vertices. The resulting graph has all vertices of even degree, because we added one edge incident to each vertex formerly of odd degree. Therefore it can be made from one balloon forming a loop. Now we remove all the edges we added. The number of removed edges, and hence the number of resulting balloons, is half the number of odd-degree vertices.

It now becomes an easy exercise to figure out how many balloons we need for our favorite polyhedron. For example, a tetrahedron requires only two balloons, as shown in Figure 7; see also [EKR00]. An icosahedron requires six balloons, which can be made into identical two-triangle units as shown in Figure 8. The same unit can make a snub cube from twelve balloons, as shown in Figure 9, as well as a snub dodecahedron from thirty balloons (an exercise: see also [Sab08]). Note that, in both cases, there are two fundamentally different ways in how the balloons can be assembled, right-handed and left-handed. This is a nice lesson on chirality (handedness).

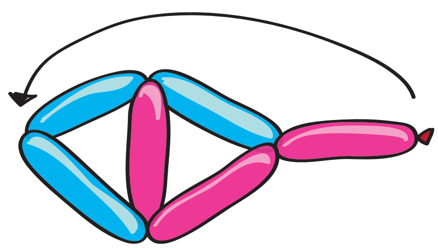

Figure 7: The tetrahedron is easy to twist from two balloons.

For all Platonic and Archimedean solids, there is a construction out of balloons that (1) uses the fewest possible balloons, (2) uses balloons all of the same length, and (3) preserves all or most of the symmetry of the polyhedron [DDH08]. Achieving all of these properties together can be a fun puzzle. In fact, achieving just the first two properties is a computationally difficult problem: decomposing a graph into a desired number of equal-length balloons is a special case of "Holyer's problem" in graph theory, and it turns out to be among the family of difficult "NP-complete" problems, even for making polyhedra [DDH08, Theorem 6].

5 Tangles

Balloons can make many more graphs than just polyhedra. A simple extension is to look at disconnected graphs, made from multiple shapes. Alan Holden's book *Orderly Tangles* [Hol83] introduced *regular polylinks*, symmetric arrangements of identical regular polygons, which make a good subject for balloon twisting. Tangles have recently been explored with the aid of (freely available) computer software to find the right thicknesses of the pieces [Har09], which may be especially useful for determining the best-size balloons for twisting.

Perhaps the simplest example of a tangle is a model of the Borromean rings assembled from three rectangles, each lying in a coordinate plane. This model is fairly easy to construct from three balloons as shown in Figure 10; see [Wor07] for a variation. Figure 11 shows a more challenging construction made from six squares; the difficulty is not twisting the squares, of course, but inter-linking them in the correct over-under pattern. Harder still are the six-pentagon tangle and the

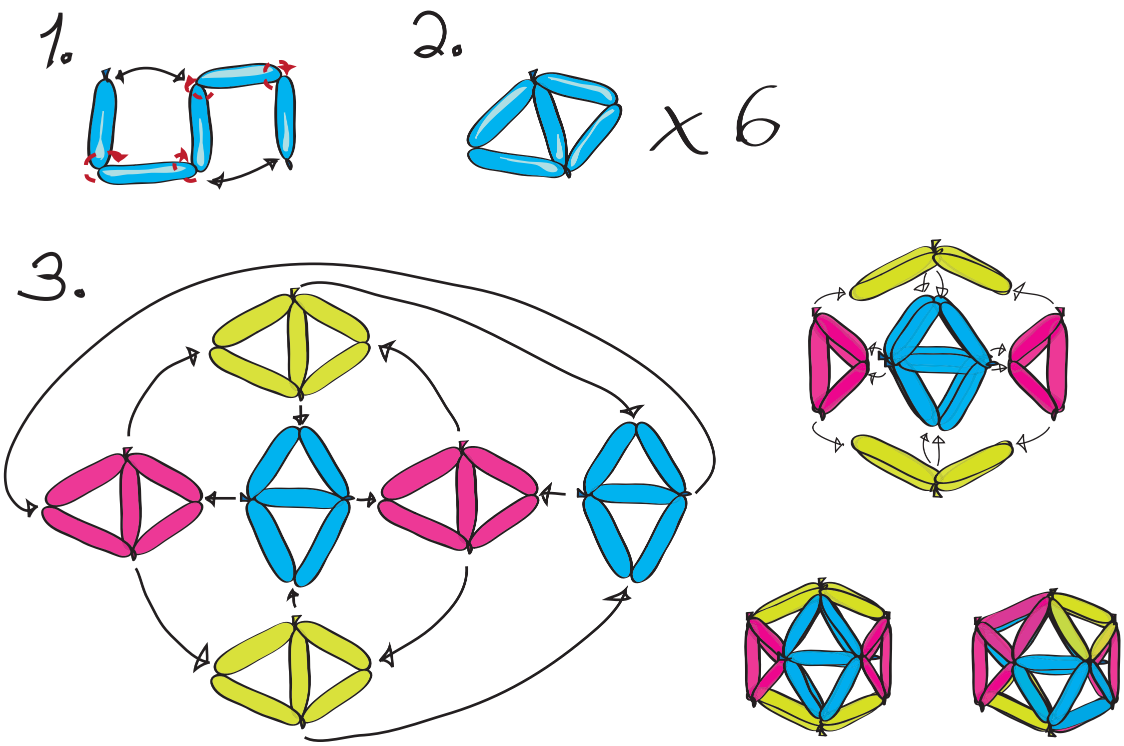

Figure 8: The icosahedron is a good example of joining several (six) identical balloon units. The tied balloon ends make it easy to attach vertices together.

four-triangle tangle; see [Har09].

A further generalization of regular polylinks are *polypolyhedra* [Lan02], which allow the symmetrically arranged shapes to be Platonic solids in addition to regular polygons. Some examples such as the famous "five intersecting tetrahedra" have been around for many years, and recently even the subject of balloon twisting [Wor07, DDH08]. But a thorough enumeration of all polypolyhedra is relatively recent [Lan02]. For a longer project, ideally with a group of mathematical balloon twisters, we recommend looking through this catalog of polypolyhedra.

6 Practical Guide for Twisting Balloon Polyhedra

Twisting balloons into polyhedra and related structures has been explored by several others in recent years [EKR00, Wor07, Sha07, Mor07, Sab08]. In particular, you may be interested in video instructions [Mor07] and in chemical 3D tilings [EKR00]. In the rest of this section, we give some practical tips for twisting your own balloons into polyhedra.

Long skinny balloons come in two main sizes: 160s (1-inch diameter, 60-inch length) and 260s (2 inches by 60 inches). For making one-balloon polyhedra, or complicated tangles, 160s are better. For most multiple-balloon polyhedra, the extra thickness of the 260s is better for stability. Of course, by not inflating all the way, or by attaching multiple balloons together, the balloon can reach any width-to-length ratio you like.

One of the biggest challenges is to get the balloon segments to be of equal length. Twisting a single balloon into the correct number of equal-length segments is something that just takes practice. If your lengths are off, you can always untwist and start again. When working with

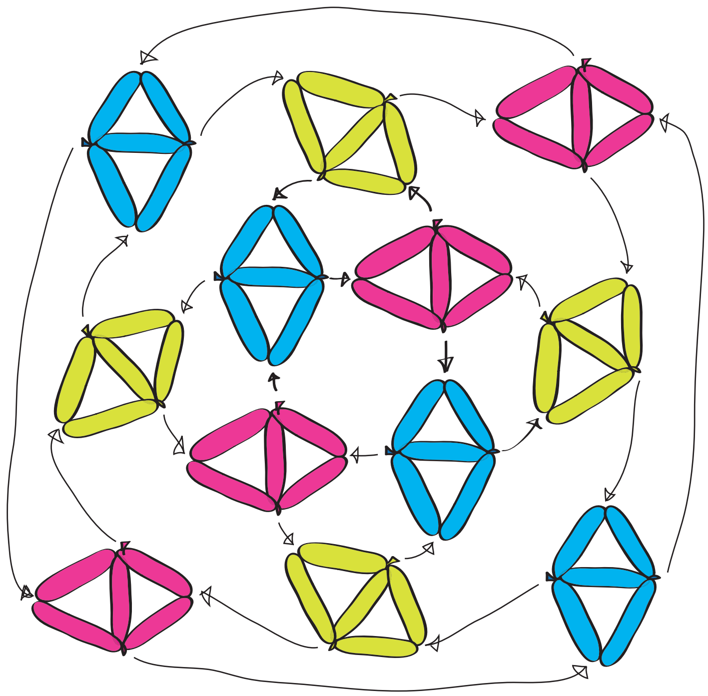

Figure 9: The snub cube is a bigger example of joining several (twelve) identical balloon units, recommended for groups of polyhedral balloon twisters. We leave the final form as a surprise.

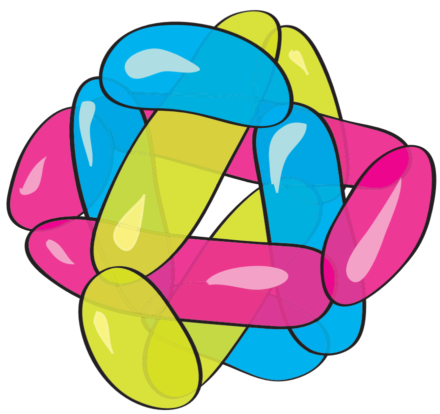

Figure 10: The Borromean rings are easy to make from three balloons.

multiple balloons, it can be helpful to inflate them all to the same length before twisting them.

You may want to shorten the lengths of all the edges on a finished balloon structure, for example to get a tangle to fit together snugly. One way to do this is to grab all the edges at a vertex, and twist them all together. Performing such a twist at every vertex results in an aesthetically pleasing effect that accents the vertices.

Another problem you may run into is wanting to pass the end of a balloon through a hole that is narrower than the inflated balloon. If you can fit the deflated end of the balloon through, you can then squeeze the air through the deflated portion and inflate the other side. It is better, though, to figure out a twisting order that avoids this, and this is a puzzle in itself.

Enjoy your mathematical twists!

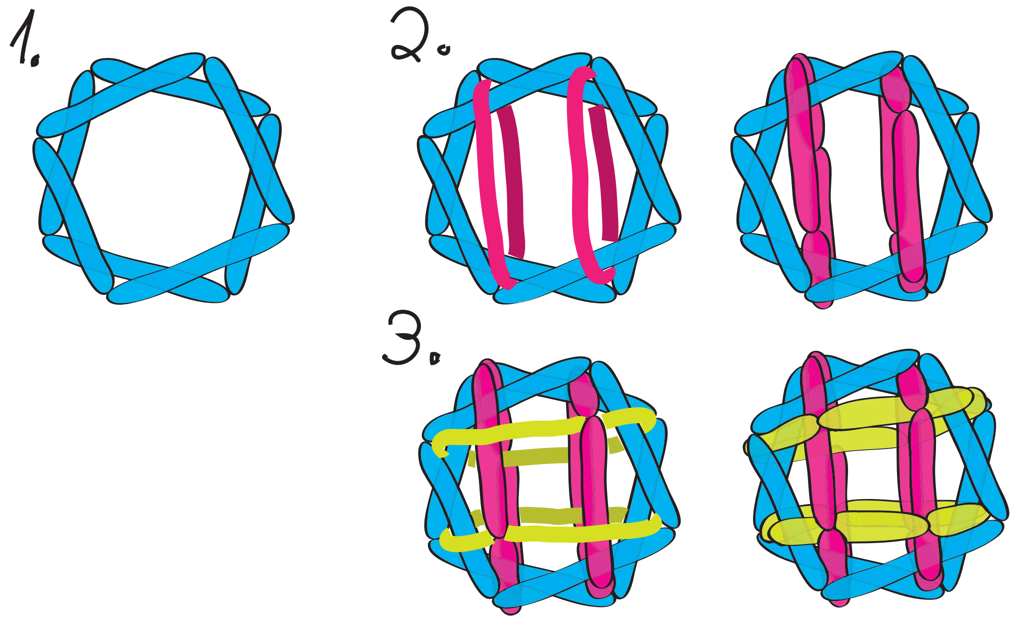

Figure 11: The six-square tangle is a good puzzle in assembling balloon tangles.

References

[BM76] J. A. Bondy and U. S. R. Murty. *Graph Theory with Applications*. North-Holland, 1976.

[DDH08] Erik D. Demaine, Martin L. Demaine, and Vi Hart. Computational balloon twisting: The theory of balloon polyhedra. In *Proceedings of the 20th Canadian Conference on Computational Geometry*, Montréal, Canada, August 2008.

[EKR00] Rolf Eckhardt, Asif Karim, and Marcus Rehbein. Balloon molecules. http://www.balloonmolecules.com/, teaching molecular models from balloons since 2000. First described in *Nachrichten aus der Chemie* 48:1541–1542, December 2000. The German website, http://www.ballonmolekuele.de/, has more information including actual constructions.

[Eul36] Leonh. Eulero. Solutio problematis ad geometriam situs pertinentis. *Commentarii academiae scientiarum Petropolitanae*, 8:128–140, 1736.

[GJ79] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., 1979.

[Har09] George W. Hart. Orderly tangles revisited. In *Mathematical Wizardry for a Gardner*, pages 187–210, 2009. http://www.georgehart.com/orderly-tangles-revisited/tangles.htm.

[Hol83] Alan Holden. *Orderly Tangles: Cloverleafs, Gordian Knots, and Regular Polylinks*. Columbia University Press, 1983.

[Lan02] Robert J. Lang. Polypolyhedra in origami. In *Origami^3: Proceedings of the 3rd International Meeting of Origami Science, Math, and Education*, pages 153–168, 2002. http://www.langorigami.com/science/polypolyhedra/polypolyhedra.php4.

[Mor07] Jim Morey. YouTube: theAmatour's channel. http://www.youtube.com/user/theAmatour, instructional videos of balloon polyhedra since November 6, 2007. See also http://moria.wesleyancollege.edu/faculty/morey/.

[Sab08] Mishel Sabbah. Polyhedrons: Snub dodecahedron and snub truncated icosahedron. Balloon Animal Forum posting, March 7, 2008. http://www.balloon-animals.com/forum/index.php?topic=931.0. See also the YouTube video http://www.youtube.com/watch?v=Hf4tfoZC_L4.

[Sha07] Jeremy Shafer. Icosahedron balloon ball. *Bay Area Rapid Folders Newsletter*, Summer/Fall 2007.

[Wor07] Carl D. Worth. Balloon twisting. http://www.cworth.org/balloon_twisting/, June 5, 2007.